Math 331 - Ordinary Differential Equations

October 21, 2020

Mathematical Models and Direction Fields

Definition: A **Differential Equation** is an equation that contains a derivative

Definition: A **Mathematical Model** is an equation that describes a physical application, along with the definitions of the variables used

One of the most common application of Differential Equations involves Newton's Second Law: $F = ma$.

It is often the case that we can measure the forces acting on an object, along with it's mass. Which leaves the acceleration, a, as the only unknown.

Since a is a derivative, in particular $a = \frac{dv}{dt}$, this gives us a differential equation.

Example: Suppose that an object is falling. The forces that we will consider to be acting on this object are gravity and air resistance.

From Newton's Second Law, we have on the one hand that:

$$mg - \gamma v = F = ma = m\frac{dv}{dt}$$

The force due to air resistance is proportional to the velocity, v. That is the force due to air resistance can be written as $-\gamma v$, where γ is the drag coefficient. Note that this is negative since the force of air resistance is acting opposite the force of gravity.

So, we have the differential equation:

$$m\frac{dv}{dt} = mg - \gamma v$$

Mathematical Models and Direction Fields

Example: Suppose an object with mass $m = 10$ and drag coefficient $\gamma = 2$ is falling. Then our equation becomes

$$10\frac{dv}{dt} = 10 \cdot 9.8 - 2v$$

Which can be reduced, by dividing by 10, to:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

Let's try to see what information we can get from the Differential Equation, before we try to solve it.

Since the derivative $\frac{dv}{dt}$ gives us the slope of v in the $vt-$plane, we can use the differential equation to compute the slope of v at various points.

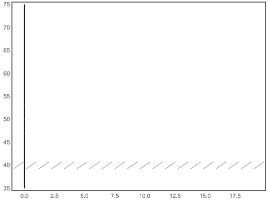
Using $v = 40$, we find $\frac{dv}{dt} = 9.8 - \frac{40}{5} = 9.8 - 8 = 1.8$.

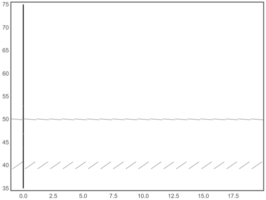
Plotting this in the $vt-$plane, we can see the slopes of v when $v = 40$

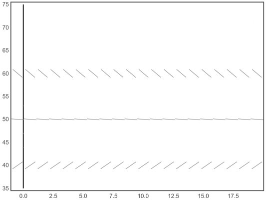
The plot of slopes of v in the $vt-$plane is called a **slope field**, or **direction field**.

Mathematical Models and Direction Fields

The plot of the slopes of v in the $vt-$plane when $v = 40$:







As we compute and plot $\frac{dv}{dt}$ for more values of v, we get a fuller view of $v(t)$

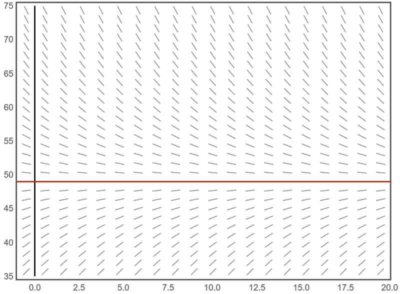
For $v = 50$, we compute $\frac{dv}{dt} = 9.8 - \frac{50}{5} = 9.8 - 10 = -.2$

For $v = 60$, we compute $\frac{dv}{dt} = 9.8 - \frac{60}{5} = 9.8 - 12 = -2.2$

While we wouldn't want to compute all of these by hand, we can use computers to compute 100 or more slopes for a slope field very quickly.

Mathematical Models and Direction Fields

Let's look at the direction field with more vectors computed.



Notice that at one $v-$value, the tangent vectors are flat.
What is happening at this red line?
Here all of the values of $\frac{dv}{dt} = 0$. Above it, $\frac{dv}{dt} < 0$ and below it, $\frac{dv}{dt} > 0$
To find the $v-$value here, we solve our differential equation for $\frac{dv}{dt} = 0$

$$0 = \frac{dv}{dt} = 9.8 - \frac{v}{5}$$

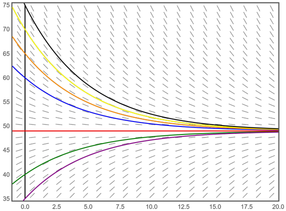
Adding $\frac{v}{5}$ and multiplying by 5 on both sides, yields $v = 49$
At $v = 49$, $\frac{dv}{dt} = 0$. We call this an **equilibirum solution**
Note that when something is in equilibrium, it is not changing - that is, the rate of change (derivative) is 0.

Equilibrium Solutions of Differential Equations

Example: We modeled the velocity of a falling object with a Diff. Eq.

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

We haven't found formulas for solutions, but we learned a lot about the sol'ns.
We created a direction field, which shows tangent vectors to solutions.



We even found one solution, called the equilibrium solution, at $v = 49$
We can, also, use the direction field to draw in other solutions.
Solutions are drawn so each vector it touches is tangent to the solution curve.
Since $\frac{dv}{dt} > 0$ for $v < 49$ the solution curve increases up towards $v = 49$
Similarly, solutions with $v > 49$ satisfy $\frac{dv}{dt} < 0$ and thus the solution curves decrease down towards $v = 49$.
Note 1: ALL solutions tend towards $v = 49$ as $t \to \infty$
Note 2: We were able to draw this conclusion just by analyzing the eq. sol.

Equilibrium Solutions of Differential Equations

Example: Find the equilibrium solution of

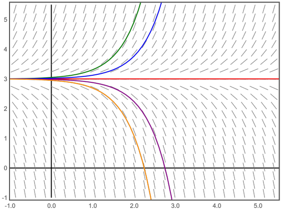
$$\frac{dy}{dt} = 2y - 6$$

and analyze the equilibrium solution to find $\lim_{t\to\infty} y$

The equilibrium solution occurs when $\frac{dy}{dt} = 0$.

Solving $2y - 6 = \frac{dy}{dt} = 0$ for y, we get the eq. sol. $y = 3$

We can, also, see this equilibrium solution in the direction field



Notice here, though, that other solutions diverge away from this eq. sol.

That is, if $y > 3$ then $\frac{dy}{dt} > 0$ and solutions satisfy: $\lim_{t\to\infty} y = \infty$

If $y < 3$ then $\frac{dy}{dt} < 0$ and solutions satisfy: $\lim_{t\to\infty} y = -\infty$

We will further study and classify eq. sol. later in the semester.

Mathematical Models - Pond Example

Example: Consider a pond that holds $10,000m^3$ of water that has one stream running into it and on running out. Water from Stream A flows in at $500m^3/day$ while Stream B flows out at $500m^3/day$, so the amount of water stays constant. At time $t = 0$ Stream A becomes contaminated with road salt at a concentration of $5kg/1000m^3$. Assume that the contaminate is evenly mixed throughout the pond.

Let $S(t)$ be the amount of salt in the pond after t days of pollution, find $S(t)$.

Solution: To find $S(t)$, we will need to model $S(t)$ using a Diff. Eq.

To do this, we will want to look at how $S(t)$ changes, distinguishing between the rate at which salt is coming in and going out.

Rate in $= \frac{5kg}{1000\cancel{m^3}} \cdot \frac{500\cancel{m^3}}{day} = \frac{2500kg}{1000days} = \frac{2.5kg}{day}$

Rate out $=$ Concentration $\cdot \frac{500m^3}{day} = \frac{Skg}{10000\cancel{m^3}} \cdot \frac{500\cancel{m^3}}{day} = \frac{500S}{10000} \frac{kg}{day} = \frac{S}{20} \frac{kg}{day}$

Combining these, we get: $\frac{dS}{dt} =$ Rate in - Rate out $= 2.5 - \frac{S}{20}$

This leaves the differential equation:

$$\frac{dS}{dt} = 2.5 - \frac{S}{20}$$

Mathematical Models - Pond Example

To find $S(t)$, we will need to solve the differential equation:

$$\frac{dS}{dt} = 2.5 - \frac{S}{20}$$

To start, we can factor $-\frac{1}{20}$ out on the right hand side, leaving:

$$\frac{dS}{dt} = -\frac{1}{20}\left(-50 + S\right)$$

Dividing by $(S - 50)$ on both sides to use the substitution rule, we get:

$$\frac{1}{S-50}\frac{dS}{dt} = -\frac{1}{20}$$

Integrating both sides, we get:

$$\ln|S - 50| = \int \frac{1}{S-50}dS = \int \frac{1}{S-50}\frac{dS}{dt}dt = \int -\frac{1}{20}dt = -\frac{t}{20} + c$$

Exponentiating both sides gives: $|S - 50| = e^{\left(-\frac{t}{20}+c\right)} = e^c \cdot e^{\left(-\frac{t}{20}\right)}$

And thus: $S - 50 = \pm e^c \cdot e^{\left(-\frac{t}{20}\right)}$

Letting k be the constant $k = \pm e^c$, we get the solution: $S(t) = 50 + k \cdot e^{\left(-\frac{t}{20}\right)}$

Note: Regardless of the value of k, the term $k \cdot e^{\left(-\frac{t}{20}\right)} \to 0$ as $t \to \infty$

So, in the long run, $\lim_{t\to\infty} S(t) = 50$ for all values of k.

Solutions of Differential Equations

In Calc 2, we studied equations of the form:

$\frac{dv}{dt} = f(t)$ such as $\frac{dv}{dt} = 2t$

and found the family of antiderivatives, such as $v(t) = t^2 + c$ in the example.

While we didn't call it this at the time, these are differential equations.

In this course we will study differential equations of the form:

$$\frac{dv}{dt} = f(v, t)$$

That are functions of both the dependent and independent variable.

Since differential equations and integration are both rooted in the process of using information about a derivative to find information about the original function, integration will be involved in many of our techniques to solve differential equations.

Recall(Calc 2): The **substitution rule** says that:

$$\int f(v)\frac{dv}{dt}dt = \int f(v)dv$$

This will be key for our first method of solving differential equations

Solutions of Differential Equations

Recall(Calc 2): The **substitution rule** says that: $\int f(v)\frac{dv}{dt}dt = \int f(v)dv$

Example: Solve the Differential Equation $\frac{dv}{dt} = v$

Solution: In order to use the substitution rule, we need to write the left hand side as $f(v) \cdot \frac{dv}{dt}$ for some function $f(v)$.

We can attain this by multiplying both sides by $\frac{1}{v}$ to get:

$$ln|v| + C_2 = \int \frac{1}{v} dv = \int \frac{1}{v}\frac{dv}{dt}dt = \int 1dt = t + C_1$$

Now that we can use the substitution rule of the left, we can integrate both sides with respect to t.

We will start on the right, which easily integrates to: $\int 1dt = t + C$

Using the substitution rule on the left side gives $\int \frac{1}{v}\frac{dv}{dt}dt = \int \frac{1}{v}dv$

This integrates to $\int \frac{1}{v}dv = ln|v| + C$

Combining the two ends of the equations, gives us that:

$$ln|v| + C_2 = 1 + C_1$$

Since C_1 and C_2 are both constants, we can simplify this to:

$$ln|v| = t + C \qquad \text{where } C = C_1 - C_2$$

Solutions of Differential Equations

Recall(Calc 2): The **substitution rule** says that: $\int f(v)\frac{dv}{dt}dt = \int f(v)dv$

Example: Solve the Differential Equation $\frac{dv}{dt} = v$

$$ln|v| = t + C \qquad \text{where } C = C_1 - C_2$$

So, we have reduced our problem of solving a differential equation down to solving an algebraic equation, for v

Starting the algebraic process to solve for v, we exponentiate both sides to get:

$$|v| = e^{(ln|v|)} = e^{(t+C)} = e^C \cdot e^t$$

The left side reduces to just $|v|$, which is why we exponentiated

The right side can be split using our exponent rules as: $e^{(t+C)} = e^C \cdot e^t$

This means that v, itself, is: $v = \pm e^C \cdot e^t$

Since C is a constant, $k = \pm e^C$ is a constant as well.

So, we can conclude that the solutions to the diff. eq. are:

$$v = k \cdot e^t \qquad k\text{-constant}$$

Solutions of Differential Equations - revisit Ex 1

Recall(Calc 2): The **substitution rule** says that: $\int f(v)\frac{dv}{dt}dt = \int f(v)dv$

Example: Earlier, we modeled the velocity of a falling object with a Diff. Eq.

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

Find solutions for the velocity of the object, $v(t)$

Solution: In order to use the **substitution rule** to integrate the left hand side, we need it written as a function $f(v)$ *times* $\frac{dv}{dt}$

We can combine the RHS into one term (using a common denominator):

$$\frac{dv}{dt} = \frac{49-v}{5}$$

then dividing both sides by $(49 - v)$

This gives us:

$$\frac{1}{49-v}\frac{dv}{dt} = \frac{1}{5}$$

Writing it in this form, we can integrate both sides w.r.t. t:

$$\int \frac{1}{49-v}\frac{dv}{dt}dt = \int \frac{1}{5}dt$$

Solutions of Differential Equations - revisit Ex 1

$$\int \frac{1}{49-v}\frac{dv}{dt}dt = \int \frac{1}{5}dt$$

On the right hand side, we can directly integrate to get $\frac{t}{5} + C$
On the left hand side, we can use the substitution rule to get:

$$\int \frac{1}{49-v}\frac{dv}{dt}dt = \int \frac{1}{49-v}dv$$

Which integrates to $-ln|49-v|$
So, we have that the solutions of our differential equation satisfy:

$$-ln|49-v| = \frac{t}{5} + C$$

Solving for v, we multiply by (-1) and exponentiate both sides to get:

$$|49-v| = e^{-\left(\frac{t}{5}+C\right)} = e^{-C} \cdot e^{-\left(\frac{t}{5}\right)}$$

Which can be written as:

$$49 - v = \pm e^{-C} \cdot e^{-\left(\frac{t}{5}\right)} = c \cdot e^{-\left(\frac{t}{5}\right)}$$

where $c = \pm e^{-C}$
So, we can conclude that $v = 49 - c \cdot e^{-\left(\frac{t}{5}\right)}$
Since, c is an arbitrary constant, we can simplify this to:

$$v = 49 + k \cdot e^{-\left(\frac{t}{5}\right)}$$

where $k = -c$

Solutions of Differential Equations - Ex 2

In a basic population model of a population, $p(t)$, the growth rate, $\frac{dp}{dt}$, is proportional to the population.

That is, it can be modeled by the diff. eq.: $\frac{dp}{dt} = \gamma p$ for a constant γ

Example: The population of Mice, $p(t)$, in a field t months after inital measurements are taken can be modeled by:

$$\frac{dp}{dt} = \frac{p}{2} - 450$$

Let's consider, further, that there are owls eating 450 mice per month.

This introduces a negative term in the differential equation model.

Find solutions for the population of mice, $p(t)$

Solution: In order to use the **substitution rule** to integrate the left hand side, we need it written as a function $f(p)$ *times* $\frac{dp}{dt}$

We can combine the RHS into one term (using a common denominator):

$$\frac{dp}{dt} = \frac{p-900}{2}$$

then dividing both sides by $(p - 900)$

This gives us:

$$\frac{1}{p-900}\frac{dp}{dt} = \frac{1}{2}$$

Writing it in this form, we can integrate both sides w.r.t. t:

$$\int \frac{1}{p-900}\frac{dp}{dt}dt = \int \frac{1}{2}dt$$

Recall(Calc 2): The **substitution rule** says that: $\int f(v)\frac{dv}{dt}dt = \int f(v)dv$

Solutions of Differential Equations - Ex 2

$$\int \frac{1}{p-900}\frac{dp}{dt}dt = \int \frac{1}{2}dt$$

On the right hand side, we can directly integrate to get $\frac{t}{2}+C$
On the left hand side, we can use the substitution rule to get:

$$\int \frac{1}{p-900}\frac{dp}{dt}dt = \int \frac{1}{p-900}dp$$

Which integrates to $ln|p-900|$
So, we have that the solutions of our differential equation satisfy:

$$ln|p-900| = \frac{t}{2}+C$$

Now, to solve this for p, we need to exponentiate both sides to get:

$$|p-900| = e^{\left(\frac{t}{2}\right)+C} = e^{C}\cdot e^{\left(\frac{t}{2}\right)}$$

Which can be written as:

$$p-900 = \pm e^{C}\cdot e^{\left(\frac{t}{2}\right)} = k\cdot e^{\left(\frac{t}{2}\right)}$$

where $k=\pm e^{C}$
So, we can conclude that $p = 900 + k\cdot e^{\left(\frac{t}{2}\right)}$

Solutions of Diff. Eq. - Ex 2 with Initial Condition

Let's revisit our population model for mice that we saw in the last example.

Example: The population of Mice, $p(t)$, in a field t months after inital measurements are taken can be modeled by:

$$\frac{dp}{dt} = \frac{p}{2} - 450$$

We found the solution to be given by: $p = 900 + k \cdot e^{\left(\frac{t}{2}\right)}$

What if we want to use our model to predict the number of mice after 1 year?

In terms of our variables, this is asking: What is $p(12)$?

Because of the unknown constant k, we are unable to make this prediction.

This is because our model is based on the rate of change, $\frac{dp}{dt}$.

Similar to what we learned in integral calculus, we can only predict the change in our function from the rate of change.

To fully find $p(t)$, we need some information about $p(t)$.

It is sufficient to know just one data point, $p(t_o) = p_o$

Such a data point about $p(t)$ is called an initial condition.

Solutions of Diff. Eq. - Ex 2 with Initial Condition

Example: The population of Mice, $p(t)$, in a field t months after inital measurements are taken can be modeled by:

$$\frac{dp}{dt} = \frac{p}{2} - 450$$

Suppose the initial population at time $t = 0$, is given by the initial condition:

$$p(0) = 905$$

Find the model's prediction for the number of mice in the field after 1 year.

Using our initial condition with the family of solutions: $p(t) = 900 + k \cdot e^{\left(\frac{t}{2}\right)}$
We can determine k.
In particular, we have that:

$$905 = p(0) = 900 + k \cdot e^{\left(\frac{0}{2}\right)} = 900 + k$$

Which we can solve by subtracting 900 to get: $k = 5$

This leaves us with the solution: $p(t) = 900 + 5 \cdot e^{\left(\frac{t}{2}\right)}$

Which we can now evaluate at $t = 12$ to find: $p(12) = 900 + 5 \cdot e^{\left(\frac{12}{2}\right)} \approx 2917$
This model predicts that after one year, the pop. will grow to 2917 mice.
Note: Later in the course, we will improve on this model to add a maximum sustainable population, rather than the unbounded growth given by this exponential function.

Types of Differential Equations

Similar to Integration in Calculus 2, we will learn techniques to solve Differential Equations

So, recognizing different types of differential equations will be helpful in recognizing which technique to use

All differential equations that we've focused on so far, and all that we will cover in this course are **ordinary differntial equations**

An **ordinary differential equation** is a differential equation which uses ordinary derivatives

In contrast, a **partial differential equation** is a differential equation which uses partial derivatives

Example: A classical partial differential equation is the Heat Equation:

$$\alpha^2 \frac{\delta^2 u(x,t)}{\delta x^2} = \frac{\delta u(x,t)}{\delta t}$$

where α is a constant and u is the heat on a wire at position x and time t

In future courses, you may study partial differential equations

Types of Differential Equations

In some applications we will have two unknown functions of the same variable t

A classical example of this in ecology are predator-prey models

Recall(Lecture 1): Field Mice and Owls We may want to revisit our owl and field mice model to factor in what would happen to the owl population, as it would depend on the field mice

When we have differential equations for two different functions that depend on a single variable (and each other) then we call this a **system of differential equations**

Example:

$$\frac{dx}{dt} = ax - \alpha xy$$
$$\frac{dy}{dt} = -cx + \gamma xy$$

If x is the mouse population and y the owl population, then in this model the owl population benefits from owl-mouse interactions (the xy term) with a positive constant γ in its differential equation. Conversely, the mouse population is negatively impacted by these interactions, so there is a negative constant $-\alpha$ of the interactions of the differential equation for x.

We will study systems of differential equations later in the course

Types of Differential Equations

The **order** of a differential equation gives the highest derivative that arises in the differential equation

Example: What is the order of the differential equation below?

$$p''' + 2tp'' + e^{5t}p' + p^5 = t^7$$

Since the highest derivative involved is the third derivative, p'''

So, we can conclude that this is a differential equation of **order 3**

Types of Differential Equations

The first type of differential that we will learn to solve is called a **linear** differential equation.

The examples we studied so far are examples of linear diff. eq.

Suppose we have a differential equation of $y = y(t)$ written as:

$$F(t, y, y', ..., y^{(n)}) = 0$$

We say that y is a **linear** differential equation if F is linear in y, y', ..., $y^{(n)}$

Another way of saying this is that we can write the differential equation as:

$$a_o(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots a_n(t)y = g(t)$$

If a differential is not linear in y, y', ..., $y^{(n)}$ then it is called **nonlinear**.

Notice that there is no condition that each a_i is linear. The differential equation could be non-linear in t and still considered a linear differential equation.

Types of Differential Equations

A **solution** to a differential equation

$$F(t, y, y', ..., y^{(n)}) = 0$$

on the interval $a < t < b$ is a function $y = s(t)$ that makes the diff. eq. true:

$$F(t, s, s', ..., s^{(n)}) = 0$$

It is relatively easy to check if a function is a solution to a differential equation.

Example: Check that $y_1 = cos(t)$ is a solution to:

$$y'' + y = 0$$

$y_1' = -sin(t)$

$y_1'' = -cos(t)$

Since $y_1'' = -cos(t)$, we can see that $-cos(t) + cos(t) = 0$

Note that this is not a method to solve differential equations, but exemplify that it is easy to check our solutions once we find them.

Solving Linear Diff Eq with Integrating Factors

So far, we've used the substitution rule from integration to find solutions to our differential equations. The differential equations we studied there were linear, first-order differential equations with constant coefficients.

We will next look at solving linear, first order differential equations where the coefficients are not necessarily constant.

Since a differential equation must only be linear in y to be a linear differntial equation, our generic way of writing a linear, first-order diff. eq. is:

$$\frac{dy}{dt} + p(t) \cdot y = g(t)$$

Notice that we have no conditions on the functions $p(t)$ and $g(t)$

Our next method of solving linear, first-order differential equations, we will use the product rule for derivatives.

Recall(Calc 1): The product rule says:

$$\frac{d}{dt}\left(\mu(t) \cdot y(t)\right) = \mu(t) \cdot \frac{dy}{dt} + \frac{d\mu}{dt} \cdot y(t)$$

Solving Linear Diff Eq with Integrating Factors

$$\frac{dy}{dt} + p(t) \cdot y = g(t)$$

Recall(Calc 1): The product rule says:

$$\frac{d}{dt}\left(\mu(t) \cdot y(t)\right) = \mu(t) \cdot \frac{dy}{dt} + \frac{d\mu}{dt} \cdot y(t)$$

If we multiply our differential equation, above, by $\mu(t)$ we get:

$$\mu(t) \cdot \frac{dy}{dt} + \mu(t) \cdot p(t) \cdot y(t) = \mu(t) \cdot g(t)$$

Notice the similarities between the expressions on the left hand side of this equation and the right hand side of the product rule.

They both start out with $\mu(t) \cdot \frac{dy}{dt}$

Followed by a function of t *times* y

The difference in the expressions is the function of t coefficient of $y(t)$

So, these will be the same so long as $\frac{d\mu}{dt} = \mu(t) \cdot p(t)$

Solving Linear Diff Eq with Integrating Factors

$$\frac{dy}{dt} + p(t) \cdot y = g(t)$$

We conclude that:

$\frac{d}{dt}(\mu(t) \cdot y(t)) = \mu(t) \cdot \frac{dy}{dt} + \frac{d\mu}{dt} \cdot y(t) = \mu(t) \cdot \frac{dy}{dt} + \mu(t) \cdot p(t) \cdot y(t) = \mu(t) \cdot g(t)$

So long as $\frac{d\mu}{dt} = \mu(t) \cdot p(t)$

How do we find such a μ? And so what if that's true?

Notice, we can find such a μ by solving this diff. eq. of μ

$\frac{d\mu}{dt} = \mu(t) \cdot p(t) \Longrightarrow \frac{1}{\mu}\frac{d\mu}{dt} = p(t)$

Which can be solved by integrating each side to get:

Exponentiating both sides gives $\mu(t) = e^{\left(\int p(t) dt\right)}$

Solving Linear Diff Eq with Integrating Factors

$$\frac{dy}{dt} + p(t) \cdot y = g(t)$$

We conclude that:

$\frac{d}{dt}\left(\mu(t) \cdot y(t)\right) = \mu(t) \cdot \frac{dy}{dt} + \frac{d\mu}{dt} \cdot y(t) = \mu(t) \cdot \frac{dy}{dt} + \mu(t) \cdot p(t) \cdot y(t) = \mu(t) \cdot g(t)$

So long as $\mu(t) = e^{\left(\int p(t)dt\right)}$

Why is this useful?

If we pick $\mu(t)$ in such a way, then:

$\frac{d}{dt}\left(\mu(t) \cdot y(t)\right) = \mu(t) \cdot g(t)$

Integrating the left side will undo the derivative, to help us solve for $y(t)$

So, we integrate both sides to get:

$$\mu(t) \cdot y(t) = \int \frac{d}{dt}\left(\mu(t) \cdot y(t)\right) dt = \int \mu(t) \cdot g(t) dt$$

So, finally, we can conclude that:

$$y(t) = \frac{1}{\mu} \cdot \int \mu(t) \cdot g(t) dt \qquad \text{if } \mu(t) = e^{\left(\int p(t)dt\right)}$$

Integrating Factors - Example 1

$$\frac{dy}{dt} + p(t) \cdot y = g(t)$$

We can find $y(t)$ as:

$$y(t) = \frac{1}{\mu} \cdot \int \mu(t) \cdot g(t) dt \qquad \text{if } \mu(t) = e^{\left(\int p(t) dt\right)}$$

Example: Solve the Differential Equation:

$$\frac{dy}{dt} + \frac{1}{3} y = e^{t/2}$$

Solution: We need to find an integrating factor μ so that:

$\mu = e^{\left(\int 1/3 dt\right)} = e^{(t/3+c)}$

Since we just need any μ, we can take $c = 0$ so that $\mu = e^{(t/3)}$

Then, using this μ we can find $y(t)$ to be:

$y(t) = \frac{1}{e^{(t/3)}} \cdot \int e^{(t/3)} \cdot e^{t/2} dt = \frac{1}{e^{(t/3)}} \cdot \int e^{(5t/6)} dt = \frac{1}{e^{(t/3)}} \cdot \left(\frac{6}{5} e^{(5t/6)} + c\right)$

Distributing $\frac{1}{e^{(t/3)}}$, we get the general solution:

$$y = \frac{6}{5} e^{(t/2)} + \frac{c}{e^{(t/3)}}$$

Integrating Factors - Example 2

$$\frac{dy}{dt} + p(t) \cdot y = g(t)$$

We can find $y(t)$ as:

$$y(t) = \frac{1}{\mu} \cdot \int \mu \cdot g(t) dt \qquad \text{if } \mu(t) = e^{\left(\int p(t) dt\right)}$$

Example: Solve the Differential Equation:

$$\frac{dy}{dt} + \frac{2}{t} y = t - 1$$

Solution: We can see here that our differential equation has the standard form needed to solve using an integrating factor.

We can find that integrating factor to be:

$$\mu = e^{\left(\int \frac{2}{t} dt\right)} = e^{\left(2 \int \frac{1}{t} dt\right)} = e^{(2(ln|t|+c))} = \left(e^{ln|t|} \cdot e^{c}\right)^2 = |t|^2 \cdot (e^c)^2 = k \cdot t^2$$

Where $k = \pm(e^c)^2$ is a constant.

Since we just need any such μ, we can take $k = 1$ and thus $\mu = t^2$

Using this μ we get: $y = \frac{1}{t^2} \int t^2 \cdot (t-1) dt$

$$y = \frac{1}{t^2} \left(\frac{t^4}{4} - \frac{t^3}{3} + c \right)$$

Integrating Factors - Example 3

$$\frac{dy}{dt} + p(t) \cdot y = g(t)$$

We can find $y(t)$ as:

$$y(t) = \frac{1}{\mu} \cdot \int \mu(t) \cdot g(t)dt \qquad \text{if } \mu(t) = e^{\left(\int p(t)dt\right)}$$

Example: Solve the Differential Equation:

$$t\frac{dy}{dt} = t^2 - t - 2y$$

Solution: This is a linear differential equation, though it is not in standard form.

To put it in standard form, we need the terms involving $\frac{dy}{dt}$ and y on the same side of the equation.

Adding $2y$ to both sides gives: $t\frac{dy}{dt} + 2y = t^2 - t$

Additionally, to be in standard form, the coefficient of $\frac{dy}{dt}$ must be 1. We can get this by dividing by t to get:

$$\frac{dy}{dt} + \frac{2}{t}y = t - 1$$

Now that we have put it in standard form, we can see that this is the Diff. Eq. we just solved.

Integrating Factors - Application Example

Example: Consider a clean pond that holds $10,000m^3$ of water and has two streams running into it. Water from Stream A flows in at $500m^3/day$ while Stream B flows in at $750m^3/day$. Water flows out Stream C at a rate of $1250m^3/day$. At time $t = 0$ Stream A becomes contaminated with road salt at a concentration of $5kg/1000m^3$. Also at this time, someone begins dumping trash into the pond at a rate of $50m^3/day$, causing the rate of water flowing out of Stream C to increase to $1300m^3/day$.

Let $S(t)$ be the amount of salt in the pond after t days of pollution, find $S(t)$.

Solution: To find $S(t)$, we will need to model $S(t)$ using a Diff. Eq.

To do this, we will want to look at how $S(t)$ changes, distinguishing between the rate at which salt is coming in and going out.

Rate in $= \frac{5kg}{1000\cancel{m^3}} \cdot \frac{500\cancel{m^3}}{day} = \frac{2500kg}{1000days} = \frac{2.5kg}{day}$ from Stream A.

Rate out $=$ Concentration $\cdot \frac{1300m^3}{day} = \frac{Skg}{(10000-50t)\cancel{m^3}} \cdot \frac{1300\cancel{m^3}}{day} = \frac{1300S}{(10000-50t)} \frac{kg}{day}$

Combining these, we get: $\frac{dS}{dt} =$ Rate in - Rate out $= 2.5 - \frac{1300S}{10000-50t}$

This is a linear diff. eq. that can be written in standard form as:

$$\frac{dS}{dt} + \frac{1300}{10000-50t} \cdot S = 2.5$$

Integrating Factors - Application Example

This is a linear diff. eq. that can be written in standard form as:

$$\frac{dS}{dt} + \frac{1300}{10000-50t} \cdot S = 2.5$$

We can find the solution using the integrating factor: $\mu = e^{\left(\int \frac{1300}{10000-50t} dt\right)}$

Computing this integral, we get:
$\int \frac{1300}{10000-50t} dt = -\frac{1300}{50} \int \frac{1}{t-200} dt = -26 ln|t-200| + c$

So, we can choose an integrating factor:

$$\mu = e^{\left(\int \frac{1300}{10000-50t} dt\right)} = e^{(-26 ln|t-200|)} = \left(e^{(ln|t-200|)}\right)^{(-26)} = (t-200)^{(-26)}$$

Using this integrating factor, we get:

$$S = \frac{1}{(t-200)^{-26}} \int 2.5(t-200)^{-26} dt = 2.5(t-200)^{26} \int (t-200)^{-26} dt$$

$$S = 2.5(t-200)^{26} \cdot \left(\frac{-1}{25}(t-200)^{(-25)} + C\right) = \frac{-2.5}{25}(t-200) + C \cdot (t-200)^{26}$$

This gives the family of solutions of the differential equation.

But what value of C should we choose for this application?

Integrating Factors - Application Example

Example: **Consider a clean pond** that holds $10,000m^3$ of water and has two streams running into it. Water from Stream A flows in at $500m^3/day$ while Stream B flows in at $750m^3/day$. Water flows out Stream C at a rate of $1250m^3/day$. At time $t = 0$ Stream A becomes contaminated with road salt at a concentration of $5kg/1000m^3$. Also at this time, someone begins dumping trash into the pond at a rate of $50m^3/day$, causing the rate of water flowing out of Stream C to increase to $1300m^3/day$.

Solution: $S = \frac{-2.5}{25}(t-200) + C \cdot (t-200)^{26} = \frac{-1}{10}(t-200) + C \cdot (t-200)^{26}$
Further, we know that the pond is clean at time $t = 0$. That is: $S(0) = 0$
Using this in our formula for $S(t)$ we get:

$$0 = S(0) = \frac{-1}{10}(0-200) + C \cdot (0-200)^{26}$$

Simplifying, we get that: $0 = 20 + C \cdot 200^{26}$

Subtracting 20 from both sides and multiplying by 200^{-26} gives: $C = \dfrac{-20}{200^{26}}$

So, we can conclude that: $S(t) = \dfrac{-1}{10}(t-200) - 20 \cdot \left(\dfrac{t-200}{200}\right)^{26}$

Solving Separable Differential Equations

We learned how to solve a special class of first order differential equations - linear ones.

In this section we will study how to solve another class of first-order differential equations - **separable** first-order differential equations.

A **separable** first-order differential equation is a differential equation that can be written in the form:

$$N(y)\frac{dy}{dx} = M(x)$$

Note: Some resources will write the definition in different forms such as:

$$M(x) + N(y)\frac{dy}{dx} = 0 \quad \text{OR} \quad \frac{dy}{dx} = M(x)\cdot N(y) \quad \text{OR} \quad M(x)dx + N(y)dy = 0$$

All of these forms are equivalent up to a negative sign or an inverse.

Using the form $N(y)\frac{dy}{dx} = M(x)$, we can integrate both sides with respect to x

$$\int N(y)dy = \int N(y)\frac{dy}{dx}dx = \int M(x)dx$$

The left hand side can be simplified using the **Substitution Rule**

Solving Separable Differential Equations

If we have a separable differential equation:

$$N(y)\tfrac{dy}{dx} = M(x)$$

then we can integrate both sides to get:

$$\int N(y)dy = \int N(y)\frac{dy}{dx}dx = \int M(x)dx$$

Example: Find solutions to the Differential Equation:

$$\tfrac{dy}{dx} = \tfrac{x^2}{1-y^2}$$

Solution: To see this is separable, multiply both sides by $1 - y^2$ to get

$$\left(1 - y^2\right)\tfrac{dy}{dx} = x^2$$

So, we can conclude that the solutions y satisfy:

$$y - \frac{y^3}{3} = \frac{x^3}{3} + c$$

Solving Separable Differential Equations

The solutions to the differential equation:

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

Satisfy the equation:

$$y - \frac{y^3}{3} = \frac{x^3}{3} + c$$

Note: We cannot solve this equation explicitly for y by itself

We call this an **implicit solution** to the differential equation

Often times non-linear differential equations, like most separable equations, cannot be solved explicitly.

However, implicit solutions are often just as useful in applications because we can still compute numerical solutions and create integral curves from them.

Solving Separable Differential Equations - Example 2

Example: Find solutions to the Initial Value Problem:

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2y-2}\ , \quad y(0) = -1$$

Solution: To see this is separable, multiply both sides by $2y - 2$ to get

$$(2y-2)\,\frac{dy}{dx} = 3x^2 + 4x + 2$$

So, we can conclude that the solutions y satisfy:

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

Using the initial condition $y(0) = -1$ we have:
So, we can conclude that y can be found implicitly as:

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

Solving Separable Differential Equations - Example 2

The solutions to the differential equation:

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2y-2}$$

Satisfy the equation:

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

Notice here that we have a quadratic equation in y

$$y^2 - 2y - \left(x^3 + 2x^2 + 2x + 3\right) = 0$$

So, we can solve this for y explicitly using the quadratic equation
Doing the quadratic equation yields:

$$\begin{aligned} y &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-(x^3+2x^2+2x+3))}}{2} \\ &= \frac{2 \pm \sqrt{4(1+x^3+2x^2+2x+3)}}{2} = \frac{2 \pm 2\sqrt{x^3+2x^2+2x+4}}{2} \\ &= 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4} \end{aligned}$$

Simplifying under the square root and factoring out 4
And dividing by 2 gives a simplified version of y.
However, only one of these satisfies our initial condition $y(0) = -1$, which is:

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

So, we have found an explicit solution to our differential equation
Note: It is prefered to solve a separable diff. eq. explicitly for y if you can.
Since it is rare that we can find explicit solutions to separable differential equations, we often need to leave our solutions with y implicitly defined.

Solving Separable Differential Equations - Example 3

Our method of solving Separable Diff. Eq. relies on computing two integrals. As we've seen in calculus, some integrals are easier than others to compute. In this example, we'll need to work a little harder to compute the integral.

Example: Find solutions to the Differential Equation:

$$\frac{dv}{dx} = \frac{v^2+3v+2}{x}$$

Solution: To see this is separable, divide both sides by $v^2 + 3v + 2$ to get

$$\frac{1}{v^2+3v+2}\frac{dv}{dx} = \frac{1}{x}$$

Now, we can integrate with respect to x to get:

$$\int \left(\frac{1}{v^2+3v+2}\right) dv = \int \left(\frac{1}{v^2+3v+2}\right)\frac{dv}{dx}dx = \int \frac{1}{x}dx = \ln|x| + c$$

The left hand side is more difficult to integrate.
To integrate it, we factor the denominator and split it using partial fractions.

$$\frac{1}{v^2+3v+2} = \frac{1}{(v+1)(v+2)} = \frac{1}{v+1} - \frac{1}{v+2}$$

We can now compute the integral:

$$\int \left(\frac{1}{v^2+3v+2}\right) dv = \ln|v+1| - \ln|v+2| = \ln\left|\frac{v+1}{v+2}\right|$$

And, thus, we have: $\ln\left|\frac{v+1}{v+2}\right| = \ln|x| + c$

This is an implicit solution to our diff. eq. that we can solve explicitly.

Solving Separable Differential Equations - Example 3

The solutions to the differential equation:

$$\frac{dv}{dx} = \frac{v^2+3v+2}{x}$$

Satisfy the implicit equation:

$$\ln\left|\frac{v+1}{v+2}\right| = \ln|x| + c$$

We can solve this explicitly for v

Exponentiating both sides yields: $\frac{v+1}{v+2} = \pm e^c \cdot x = k \cdot x$

Multiplying by $(v+2)$ yields:

$$v + 1 = k \cdot x \cdot (v+2) = kxv + 2kx$$

Subtracting 1 and kvx from both sides yields:

$$(1 - kx)v = v - kxv = 2kx - 1$$

Dividing by $(1 - kx)$ gives the explicit solution:

$$v = \frac{2kx - 1}{1 - kx}$$

Homogeneous Differential Equations

In an earlier example, we used a change of variables to change a diff. eq. into a first-order, linear diff. eq.

There is a special class of differential equations, called *homogeneous* differential equations, which can be made into separable differential equation.

A differential equation given by the function:

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y)$ can be written in terms of $\frac{y}{x}$, is called *homogeneous*.

A homogeneous diff. eq. can be changed into a separable diff. eq. with the change of variables: $v = \frac{y}{x}$

Example: The following differential equation is homogeneous

$$\frac{dy}{dx} = \frac{2x^2 + 4xy + y^2}{x^2}$$

We can see this diff. eq. is homogeneous by dividing each term by the denominator x^2 to get:

$$\frac{dy}{dx} = \frac{2x^2 + 4xy + y^2}{x^2} = \frac{2x^2}{x^2} + \frac{4xy}{x^2} + \frac{y^2}{x^2} = 2 + 4\frac{y}{x} + \left(\frac{y}{x}\right)^2$$

Homogeneous Differential Equations

Example: Solve the homogeneous differential equation

$$\frac{dy}{dx} = \frac{2x^2+4xy+y^2}{x^2} = 2 + 4\frac{y}{x} + \left(\frac{y}{x}\right)^2$$

As noted, we can solve this using a change of variables: $v = \frac{y}{x}$

With this change of variables, we can write the right hand side as: $2 + 4v + v^2$

We, also, need to change the left hand side, $\frac{dy}{dx}$, to be in terms of v

To do this, we'll use that $y = x \cdot v$ and compute $\frac{dy}{dx}$ using the product rule:

$$\frac{dy}{dx} = \frac{dx}{dx} \cdot v + x \cdot \frac{dv}{dx} = v + x \cdot \frac{dv}{dx}$$

We can now write this differential equations in terms of v and x:

$$v + x \cdot \frac{dv}{dx} = 2 + 4v + v^2$$

And solving this for $\frac{dv}{dx}$ by subtracting v and dividing by x, we get:

$$\frac{dv}{dx} = \frac{2+3v+v^2}{x}$$

We solved this diff eq to find $v = \frac{2kx-1}{1-kx}$

Since our differential equation was one of y, we still need to find the function y to have a solution to the original diff. eq.

To do this, we'll use that $y = x \cdot v$ and thus:

$$y = x \cdot \frac{2kx-1}{1-kx}$$

Mathematical Modeling Revisited - Bank example

Example: The rate at which interest earned on invested money is proportional to the amount of money in the account. This proportionality constant is called the annual interest rate, r. Find the amount of money, $S(t)$, in an account t years after a deposit of S_o is made.

Note: Here, we are assuming that interest is compounded continuously. In practice interest is most typically compounded daily, monthly, or yearly.

Solution: Since the rate of change of $S(t)$ is proportional to $S(t)$, we have:

$$\frac{dS}{dt} = r \cdot S(t)$$

Dividing by S and integrating, we get the gen sol: $S(t) = ke^{(rt)}$

Imposing our initial condition, $S(0) = S_o$ gives: $S(t) = S_o e^{(rt)}$

So, the amount of money, $S(t)$, in an account earning an interest rate, r, after t years is:

$$S(t) = S_o e^{(rt)}$$

Mathematical Modeling Revisited - Bank example

Example: The amount of money, $S(t)$, in an account earning an interest rate, r, after t years is:

$$S(t) = S_o e^{(rt)}$$

If you earn an interest rate of 6% on money invested at age 22, by what multiple will your investment grow by the time you reach age 65?

Solution: The money you invest at age 22 will have $t = 43$ years to grow until you reach age 65. At an 6% interest rate, your invest at age 65 will be worth:

$$S(43) = S_o e^{(.06 \cdot 43)} \approx 13.2 \cdot S_o$$

So, we can conclude that your investment will be worth 13.2 times the original amount that you originally invest.

As a quick example, if someone inherited $\$100,000$ at age 22 and they put in an account that got 6% interest until they retired at age 65, they would have roughly \$1.3 million.

Mathematical Modeling Revisited - Bank example

Example: Let the amount of money t years after making an initial deposit of S_o, be written as $S(t)$. If the money is in an account earning an interest rate, r, then $S(t)$ can be model by the IVP:

$$\frac{dS}{dt} = rS \qquad\qquad S(0) = S_o$$

Suppose further that you make regular deposits, totaling $\$D$ per year.

How will this impact our model?

Now money is being added to the account in two ways: from interest and deposits.

$$\frac{dS}{dt} = rS + D \qquad\qquad S(0) = S_o$$

We, again, have a model that gives a linear differential equation, which we can solve to find $S(t)$.

Mathematical Modeling Revisited - Bank example

Example: Suppose that you open retirement account with 6% interest rate at age 22. You initially invest \$1000 and deposit \$6000 per year until you retire at age 65. How much money will you have in the account when you retire?
Solution: Our question can be interpreted in variable as: What is $S(43)$?
Since we are given information on $\frac{dS}{dt}$, we start by modeling $\frac{dS}{dt}$ using the interest rate, 6%, and the yearly contribution amount of \$6000.

This gives the differential equation: $\dfrac{dS}{dt} = .06S + 6000$

Additionally, we have an initial deposit of \$1000, so $S(0) = 1000$.
Note: Combined, the diff. eq. and initial condition make this an IVP.
To find $S(43)$, we begin by solving the differential equation.

In standard form, the diff. eq. is: $\dfrac{dS}{dt} - .06S = 6000$

This leads to the integrating factor $\mu = e^{\int -.06dt} = e^{(-.06t)}$; giving:

$$S(t) = \frac{1}{e^{(-.06t)}} \int e^{(-.06t)} \cdot 6000dt = 6000e^{.06t} \cdot \left(\frac{1}{-.06} e^{(-.06t)} + c \right)$$

$$1000 = S(0) = 6000e^{.06 \cdot 0} \cdot \left(\tfrac{1}{-.06} e^{(-.06 \cdot 0)} + c \right) = \tfrac{6000}{-.06} + 6000c = 6000c - 100000$$

So $c = \frac{101000}{6000} = \frac{101}{6}$ and we find:

$S(t) = 6000e^{.06t} \cdot \left(\frac{1}{-.06} e^{(-.06t)} + \frac{101}{6} \right)$ and $S(43) \approx 1232910$

Mathematical Modeling Revisited

Mathematical models are used in many disciplines outside of math to study various applications, such as the population and physical examples we've looked at.

It is often helpful to leave some constants as parameters, that can vary within the application, to best study these ideas.

By allowing certain parameters to vary, we can study their impact on an application.

As examples, we will build models using these parameters to see how they can help us analyze an application.

In practice, it is good to run experiments to test if the theoretical results found in analyzing the model accurately describe the results found in the experiment.

Though, sometimes this can be too costly, or not possible, and we need to rely on the model.

Mathematical Modeling Revisited

Example: A tank at $t = 0$ has Q_o lb of salt dissolved in 100 gal of water. Assume that water containing $1/4$ lb of salt per gallon is entering the tank at a rate of r gallons per minute and the water is leaving the tank at the same rate. Find the limiting amount, Q_L, of the amount of salt in the tank in the long run.
Solution: As in our other mixing problems:

$$\frac{dQ}{dt} = \text{rate in - rate out}$$

The rate in can be calculated in terms of our parameter r as: rate in $= \frac{1}{4}r$
And rate out is, also, in terms of r: rate out $= \frac{Q}{100}r$
So, we have the Initial Value Problem: $\frac{dQ}{dt} = \frac{r}{4} - \frac{Qr}{100}$ and $Q(0) = Q_o$
Adding $\frac{Qr}{100}$ to both sides we have: $\frac{dQ}{dt} + \frac{Qr}{100} = \frac{r}{4}$
This leads to the integrating factor: $\mu = e^{\left(\int \frac{r}{100} dt\right)} = e^{(r/100)t}$
Gen sol: $Q(t) = e^{-(r/100)t} \int \left(\frac{r}{4} \cdot e^{(r/100)t}\right) dt = \frac{re^{-(r/100)t}}{4} \left(\frac{100}{r} \cdot e^{(r/100)t} + c\right)$

$$= \frac{\cancel{r}\cancel{e^{-(r/100)t}}}{4} \cdot \frac{100}{\cancel{r}} \cdot \cancel{e^{(r/100)t}} + \frac{re^{-(r/100)t}}{4} \cdot c = 25 + ke^{(-rt/100)}$$

Imposing the I.C. $Q(0) = Q_o$ we find $k = Q_o - 25$, so the solution to the IVP is:

$$Q(t) = 25 + (Q_o - 25)e^{(-rt/100)}$$

Long term, as $t \to \infty$, we see $Q(t) \to 25$, so $Q_L = 25$ is the limiting amount.

Mathematical Modeling Revisited

Example: A tank at $t = 0$ has Q_o lb of salt dissolved in 100 gal of water. Assume that water containing $1/4$ lb of salt per gallon is entering the tank at a rate of r gallons per minute and the water is leaving the tank at the same rate. We found the limiting amount of salt in the take, Q_L, to be $Q_L = 25$.

If the amount of salt starts at twice the limiting amount, $Q_o = 2Q_L = 50$, find r so that it takes 45 minutes for $Q(t)$ to be within 2% of Q_L.

Solution: Since $Q_o = 2Q_L$ starts above Q_L, we are looking for r so $Q(45) = Q_L + 2\% \cdot Q_L = 25.5$.

That is, we want:

$$25.5 = Q(45) = 25 + (50 - 25)e^{(-r \cdot 45/100)}$$

To solve for r, we start by subtracting 25 then dividing by $(50 - 25) = 25$ on both sides to get:

$$\frac{.5}{25} = e^{(-r \cdot 45/100)}$$

Taking the natural log gives: $ln\left(\frac{.5}{25}\right) = \frac{-45r}{100}$

So, finally we can conclude that $r = \frac{-100}{45} ln\left(\frac{.5}{25}\right) \approx 8.69$

Note: This analysis would be difficult to do without our model, as we would need to run multiple experiments varying r.

Autonomous Differential Equations

Definition: A **Autonomous Differential Equation** is diff. eq. of the form:

$$\frac{dy}{dt} = f(y)$$

That is, a differential equation where $\frac{dy}{dt}$ can be written in terms of just y.

We have seen examples in Compound Interest and Modeling mouse populations

Our most basic population model says that the rate of change of a population is proportional to the population. That is:

$$\frac{dy}{dt} = ry$$

where y is the population and r is the growth rate.

This is an example of an autonomous differential equation.

For a starting population $y(0) = y_o$, we can obtain the solution:

$$y = y_o e^{rt}$$

Autonomous Differential Equations

First model: For the population model $\frac{dy}{dt} = ry$ and initial population $y(0) = y_o$ we have the solution:

$$y = y_o e^{rt}$$

The problem with this model is that the long run behavior of y is to tend towards ∞, which does not make sense in a real world application.

Natural environments can only sustain so large of a population. We call this maximum population that an environment can sustain the **carrying capacity**.

How can we modify our population model so that $\frac{dy}{dt} \approx ry$ when y is small and decreases as y approaches the carrying capacity, K?

The easiest change to our current model is to multiply by a factor of $\left(1 - \frac{y}{K}\right)$

For small values of y, $\frac{y}{K} \approx 0$ and thus $\left(1 - \frac{y}{K}\right) \approx 1$

As $y \to K$, $\frac{y}{K} \approx 1$ and thus $\left(1 - \frac{y}{K}\right) \approx 0$

This gives our new population model, called the **logistic growth model**:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right) y$$

Autonomous Differential Equations

Logistic growth model: A population with growth rate, r, and carrying capacity, K, can be modeled by:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

Let's analyze this model by looking at its equilibrium solutions.

The eq. sol. happen when $\frac{dy}{dt} = 0$, which for this diff. eq. is:

$$r\left(1 - \frac{y}{K}\right)y = 0$$

Since this is already factored, we can see that the eq. sol. happen when:

$$\left(1 - \frac{y}{K}\right) = 0 \qquad \text{OR} \qquad y = 0$$

Solving the first equations to get $y = K$, we can conclude that the equilibrium solutions occur at $y = 0, K$

Autonomous Differential Equations

Logistic growth model: A population with growth rate, r, and carrying capacity, K, can be modeled by: $\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$

Equilibrium solutions occur at $y = 0$ and $y = K$

What can we conclude about other solutions?

If $y > K$ then: $\frac{dy}{dt} = \underbrace{r}_{+}\underbrace{\left(1 - \frac{y}{K}\right)}_{-}\underbrace{y}_{+} < 0$

So, we can conclude that if $y > K$ then y is decreasing.

We can do a similar analysis for $0 < y < K$ and $y < 0$

If $0 < y < K$ then: $\frac{dy}{dt} = \underbrace{r}_{+}\underbrace{\left(1 - \frac{y}{K}\right)}_{+}\underbrace{y}_{+} > 0$

So, we can conclude that if $0 < y < K$ then y is increasing.

If $y < 0$ then: $\frac{dy}{dt} = \underbrace{r}_{+}\underbrace{\left(1 - \frac{y}{K}\right)}_{+}\underbrace{y}_{-} < 0$

So, we can conclude that if $y < 0$ then y is decreasing.

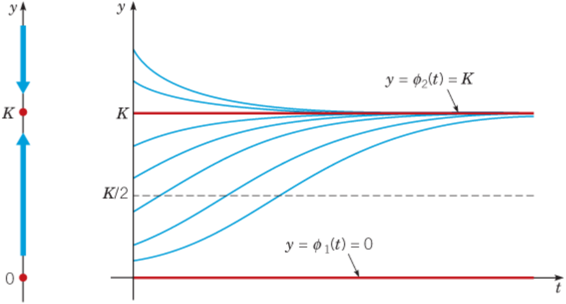
Phase Line

Autonomous Differential Equations

Logistic growth model: A population with growth rate, r, and carrying capacity, K, can be modeled by: $\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right) y$

Equilibrium solutions occur at $y = 0$ and $y = K$

If we look at the integral curves with the phase, we can draw further conclusions about our equilibrium solutions.



Since solutions that start near the equilibrium solutions $y = K$ tend toward this equilibrium solution, we call it a **asymptotically stable solution**.

Since solutions that start near the equilibrium solutions $y = 0$ diverge away from this equilibrium solution, we call it an **unstable solution**.

Autonomous Differential Equations - Example 2

Example: Find and classify the equilibrium solutions of the autonomous differential equation:

$$\frac{dy}{dt} = y(y+3)(y-2)$$

Solution: Equilibibrum solutions happen when $\frac{dy}{dt} = 0$

So, the equilibrium solutions happen when $y = 0, -3, 2$

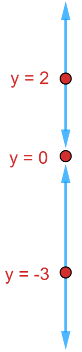
For $y > 2$, we have: $\frac{dy}{dt} = \underbrace{y}_{+}\underbrace{(y+3)}_{+}\underbrace{(y-2)}_{+} > 0$

For $0 < y < 2$, we have: $\frac{dy}{dt} = \underbrace{y}_{+}\underbrace{(y+3)}_{+}\underbrace{(y-2)}_{-} < 0$

For $-3 < y < 0$, we have: $\frac{dy}{dt} = \underbrace{y}_{-}\underbrace{(y+3)}_{+}\underbrace{(y-2)}_{-} > 0$

For $y < -3$, we have: $\frac{dy}{dt} = \underbrace{y}_{-}\underbrace{(y+3)}_{-}\underbrace{(y-2)}_{-} < 0$

Phase Line



We can see from the phase line that $y = 0$ is an asymptotically stable solution, while $y = -3$ and $y = 2$ are unstable solutions.

Autonomous Differential Equations - Example 3

Example: Find and classify the equilibrium solutions of the autonomous differential equation:

$$\frac{dy}{dt} = y^2(y+3)(y-2)$$

Solution: Equilibirum solutions happen when $\frac{dy}{dt} = 0$

So, the equilibrium solutions happen when $y = 0, -3, 2$

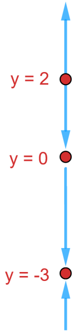
For $y > 2$, we have: $\frac{dy}{dt} = \underbrace{y^2}_{+}\underbrace{(y+3)}_{+}\underbrace{(y-2)}_{+} > 0$

For $0 < y < 2$, we have: $\frac{dy}{dt} = \underbrace{y^2}_{+}\underbrace{(y+3)}_{+}\underbrace{(y-2)}_{-} < 0$

For $-3 < y < 0$, we have: $\frac{dy}{dt} = \underbrace{y^2}_{+}\underbrace{(y+3)}_{+}\underbrace{(y-2)}_{-} < 0$

For $y < -3$, we have: $\frac{dy}{dt} = \underbrace{y^2}_{+}\underbrace{(y+3)}_{-}\underbrace{(y-2)}_{-} > 0$

Phase Line



We can see from the phase line that $y = -3$ is an asymptotically stable solution, while $y = 2$ is an unstable eq. sol.
Solutions above $y = 0$ converge to $y = 0$ while solutions below diverge away.
We call such an equilibrium solution *semi-stable.*

Existence and Uniqueness of Differential Equations

We have spent most of our work, thus far, studying and finding solutions to differential equations.

Will we always be able to find a solution?

What if we're looking for a solution that doesn't exist?

The **existence** of a solution tells us that there is a solution to be found. i.e. we're not on a wild-goose chase looking for a solution that doesn't exist.

What if we find a solution? Are there other solutions we're missing?

The **uniqueness** of a solution tells us that if we do find a solution, it's the only one!

Theorem: Existence and Uniqueness Theorem for First-Order Linear Equations
If the functions $p(t)$ and $g(t)$ are continuous on the open interval $\alpha < t < \beta$ containing $t = t_o$, then there exists a unique function $y = y_1(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for all $\alpha < t < \beta$ that satisfies an initial condition $y(t_o) = y_o$.

The proof of this relies on the existence of the integrating factor $\mu(t) = e^{\int p(t)dt}$

Existence and Uniqueness of Differential Equations

Theorem: Existence and Uniqueness Theorem for First-Order Linear Equations
If the functions $p(t)$ and $g(t)$ are continuous on the open interval $\alpha < t < \beta$ containing $t = t_o$, then there exists a unique function $y = y_1(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for all $\alpha < t < \beta$ that satisfies an initial condition $y(t_o) = y_o$.
What about Differential Equations that are non-Linear?
Theorem: Existence and Uniqueness for First-Order Differential Equations
Let the functions $f(t, y)$ and $\frac{\delta f}{\delta y}$ be continuous in some rectangle $\alpha < t < \beta$ and $\gamma < y < \delta$ containing the point (t_o, y_o). Then in some subinterval $t_o - h < t < t_o + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = y_1(t)$ of the initial value problem

$$y'(t) = f(t, y), \qquad y(t_o) = y_o$$

The proof of this theorem is left to future courses.
Important Note: The uniqueness of solutions guarantees that the graphs of two solutions of a differential equation cannot intersect.
If two solutions were to intersect at a point (a, b), then those two solutions would both solve the initial value problem with the given differential equation and initial condition $y(a) = b$, contradicting uniqueness.

Existence and Uniqueness of Differential Equations

Example: Find the solution to

$$y' = y^2, \qquad y(0) = 1$$

and find the interval where the solution exists.
Solution: Since $f(t, y) = y^2$ and $\frac{\delta f}{\delta y} = 2y$ are continuous for all reals, we know that a unique solution exists on some interval $a < t < b$
Since the diff. eq. is separable, we can solve it by integrating both sides of:

$$y^{-2}\frac{dy}{dt} = 1$$

To get: $-y^{-1} = t + c$
Which yields: $y = \frac{-1}{t+c}$
Imposing the initial condition $y(0) = 1$ we get:

$$1 = \frac{-1}{0+c}$$

So, $c = -1$ and we get the solution $y = \frac{-1}{t-1} = \frac{1}{1-t}$
This solution has a discontinuity at $t = 1$, so the largest interval on which the solution, and includes the initial data point at $t = 0$, is the interval $(-\infty, 1)$.
This interval is called *interval of existence of the solution*.

Existence and Uniqueness of Differential Equations

Our theorems tell us that unique solutions to these differential equations exist.

But they do not guarantee that we can find them!

And sometimes we cannot find solutions because the solutions cannot be written in closed form - that is solutions that can be written in terms of our usual functions.

So, what do we do if we know a function exists but we can't find it?

Numerical methods exist to find approximations to the solutions.

In practice, these approximations are as useful as the actual solution.

We will study one such method called Euler's Method soon!

Existence and Uniqueness Theorem - Non-Example

The Existence and Uniqueness theorem gives conditions on a differential equation that guarantee solutions exist and are unique.

What if these conditions aren't met?

Consider the Initial Value Problem:

$$\frac{dy}{dx} = \frac{x}{y} \text{ with } y(1) = 0$$

This is a non-linear diff. eq. that does not satisfy the conditions of the existence and uniqueness theorem since $f(x, y) = \frac{x}{y}$ is discontinuous at $y = 0$, the value at our initial condition.

Thus, there is no rectangle around the initial condition $x = 1$, $y = 0$ where $f(x, y)$ is continuous.

While the theorem does not guarantee the existence nor uniqueness of a solution, we can still explore solutions of this diff eq since it is separable.

We can see this is separable by multiplying by y to get: $y \cdot \frac{dy}{dx} = x$

Existence and Uniqueness Theorem - Non-Example

Consider the Initial Value Problem:

$$\frac{dy}{dx} = \frac{x}{y} \text{ with } y(1) = 0$$

We can see this is separable by multiplying by y to get: $y \cdot \frac{dy}{dx} = x$

Integrating both sides with respect to x gives:

$$\frac{y^2}{2} = \int y dy = \int y \frac{dy}{dx} dx = \int x dx = \frac{x^2}{2} + c$$

We can solve this for y by first multiplying by 2 to get:

$$y^2 = x^2 + k \text{ where } k = 2c$$

And then taking the square root of both sides, we yields that:

$$y = \pm\sqrt{x^2 + k}$$

At the initial condition $y(1) = 0$ we get:

$$0 = y(1) = \pm\sqrt{1^2 + k}$$

Squaring both sides gives: $0 = 1 + k$

And thus, $k = -1$

So, both $y_1 = \sqrt{x^2 - 1}$ and $y_2 = -\sqrt{x^2 - 1}$ are solutions to the differential equation, defined for $x > 1$, satisfying the initial condition $y(1) = 0$.

In this case, solutions existed despite the conditions of the theorem not being met, but the solution was not unique.

Euler's Method

We have learned how to solve several different types of differential equations.

However, there are many more differential equations out there that do not fit into one of the types we know how to solve.

While the existence and uniqueness theorems tell us that solutions exist, how do we understand the solutions to these differential equations that we do not have techniques to solve directly?

There are methods to find numerical approximation of the solutions to differential equations.

One such method, that we will look at, is called Euler's Method.

Euler's Method

Suppose that we want to find a numerical approximation of the differential equation given by:

$$\frac{dy}{dt} = f(t, y) \qquad \text{with initial condition: } y(t_o) = y_o$$

We can find the derivative, $\frac{dy}{dt}$, at (t_o, y_o) by evaluating: $\frac{dy}{dt} = f\,(t_o, y_o)$

This gives the slope of the line tangent to $y(t)$.

Using this, we can find a linear approximation to the solution, $y(t)$, near the point (t_o, y_o):

$$y(t) \approx y_o + f\,(t_o, y_o) \cdot (t - t_o)$$

Recall that this is called a *local* linear approximation.

This gives us an approximation for $y(t)$ that is useful near (t_o, y_o) but less so away from the initial condition.

How can we get an approximation for $y(t)$ that is useful away from (t_o, y_o)?

Euler's Method

Suppose that we want to find a numerical approximation of the differential equation given by:

$$\frac{dy}{dt} = f(t, y) \qquad \text{with initial condition: } y(t_o) = y_o$$

The linear approximation to the solution, $y(t)$, near (t_o, y_o) is:

$$y(t) \approx y_o + f\left(t_o, y_o\right) \cdot (t - t_o)$$

Suppose we want an approximation for $y(b)$ for some value $t = b$.
Since linear approximations have small error locally, we can use a linear approximation near t_o to estimate $y(t_1)$ for a value t_1 near t_o.
But we wanted to know $y(b)$, not $y(t_1)$. How do we get this?
We can get incrementally closer to $t = b$ by using a linear approximation of $y(t)$ at $t = t_1$ to find an approximation of $y(t_2)$ for a value t_2 near t_1.
Then a linear approximation of $y(t)$ at $t = t_2$ to find an approximation of $y(t_3)$ for a value t_3 near t_2.
Iteratively continuing this process, we can keep making approximations for t_{i+1} for a point near t_i until we reach $t_n = b$
Note: we can choose the step size, $h = \frac{b - t_o}{n}$ to ensure that $t_n = b$

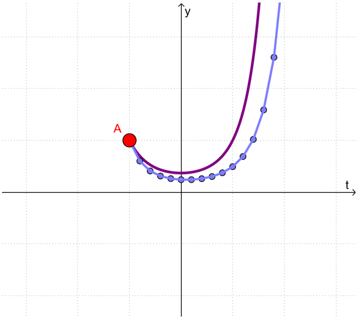
Euler's Method

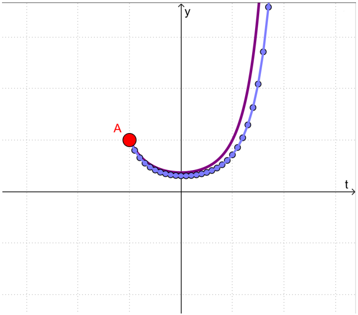
To find a numerical approximation to the Initial Value Problem:

$$\frac{dy}{dt} = f(t, y) \qquad \text{with initial condition: } y(t_o) = y_o$$

We can iteratively find a sequence of approximations with step size h:

$$t_{i+1} = t_i + h = t_o + i \cdot h$$
$$y_{i+1} = y_i + f(t_i, y_i) \cdot h$$





Note that as $h \to 0$ the approximation gets better but more steps are needed.

Euler's Method

Example: Use Euler's Method with step size $h = 0.1$ to find approximations for $t = 0.1, 0.2, \ldots, 0.9, 1$ of the solution to the initial value problem:

$$\frac{dy}{dt} = (y-1)\cdot(y-3) \qquad \text{with } y(0) = 2$$

Solution: Starting with the point $(0, 2)$, we can find an approximation for $y(0.1)$ using Euler's Mehod.

$y(0.1) \approx y_1 = 2 + f(0,2)\cdot 0.1 = 2 + (2-1)(2-3)\cdot 0.1 = 2 - .2 = 1.8$

Using this point, $(0.1, 1.8)$, we can find an approximation for $y(0.2)$ as:
$y(0.2) \approx y_2 = 1.8 + f(0.1, 1.8)\cdot 0.1 = 1.8 + (1.8-1)(1.8-3)\cdot 0.1 = 1.8 - .096 = 1.704$

Using this point, $(0.2, 1.704)$, we can find an approximation for $y(0.3)$ as:
$y(0.3) \approx y_3 = 1.704 + f(0.2, 1.704)\cdot 0.1 = 1.704 + (1.704-1)(1.704-3)\cdot 0.1 = 1.704 - .09124 = 1.61276$

Using this point, $(0.2, 1.704)$, we can find ...

This process of using one point to find the next is very tedious

Computer programs can be used to do many iterations of Euler's Method very quickly!

Exact Equations

So far, we have learned two major methods for solving differential equations.

Linear differential equations and Separable differential equations

Most differential equations, however, do not fit into either of these categories

We will learn one more method for solving a specific type of first order differential equations

For others, we will need to find numerical approximations for the solutions.

Let's start by looking at an example that we cannot solve using our current methods.

Exact Equations

Example: Solve the differential equation: $3x^2 + y^2 + 2xy\frac{dy}{dx} = 0$

This is not linear due to the y^2 term. This is, also, not separable.

So, our current methods fail.

While it is very difficult to spot, we can verify that $\psi(x, y) = x^3 + xy^2$ has the property that the partial derivatives play a defining role in our diff. eq.

In particular, $\psi_x = 3x^2 + y^2$ and $\psi_y = 2xy$ with our differential equation:

$$3x^2 + y^2 + 2xy\frac{dy}{dx} = 0$$

Since $\psi(x, y)$ is a function of x and y, but $y = y(x)$, the function $\psi(x, y(x))$ is a function of x. Thus we can compute the derivative $\frac{d\psi}{dx}$

Using the chain rule from multivariable calc, we get $\frac{d\psi}{dx} = \frac{\delta\psi}{\delta x} + \frac{\delta\psi}{\delta y}\frac{dy}{dx}$

That is, $\frac{d\psi}{dx} = \psi_x + \psi_y \cdot \frac{dy}{dx} = 3x^2 + y^2 + 2xy \cdot \frac{dy}{dx}$

But this is one side of our differential equation, so $\frac{d\psi}{dx} = 3x^2 + y^2 + 2xy \cdot \frac{dy}{dx} = 0$

And, integrating both sides with respect to x, yields

$$x^3 + xy^2 = \psi = c$$

This gives us the implicit solution $x^3 + xy^2 = c$

Exact Equations

We were able to find the implicit solution:

$$x^3 + xy^2 = c$$

to the differential equation: $3x^2 + y^2 + 2xy\frac{dy}{dx} = 0$

because we were given $\psi = x^3 + xy^2$ such that $\psi_x = 3x^2 + y^2$ and $\psi_y = 2xy$

This process can be repeated, so long as we can find such a $\psi(x, y)$

In General: A differential equation of the form:

$$M(x, y) + N(x, y) \cdot \frac{dy}{dx} = 0$$

such that $M(x, y) = \psi_x(x, y)$ and $N(x, y) = \psi_y(x, y)$ for some function $\psi(x, y)$ is called an **exact differential equation**. The implicit solution is $\psi(x, y) = c$

Exact Equations

In General: A differential equation of the form:

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such that $M(x, y) = \psi_x(x, y)$ and $N(x, y) = \psi_y(x, y)$ for some function $\psi(x, y)$ is called an **exact differential equation**. The implicit solution is $\psi(x, y) = c$

What if we weren't given $\psi(x, y)$?

How could we tell if such a ψ exists? And if ψ exists, how can we find it?

Recall(Calc 3): If $\psi(x, y)$ is twice differentiable, then $\psi_{xy} = \psi_{yx}$

So, if such a ψ exists that $M(x, y) = \psi_x(x, y)$ and $N(x, y) = \psi_y(x, y)$, then $M_y = \psi_{xy} = \psi_{yx} = N_x$

We can conclude that if $M(x, y) + N(x, y) \cdot \frac{dy}{dx} = 0$ is exact then $M_y = N_x$

What if $M_y = N_x$? Can we conclude that $M(x, y) + N(x, y) \cdot \frac{dy}{dx} = 0$ is exact?

Exact Equations

For the differential equation $M(x, y) + N(x, y) \cdot \frac{dy}{dx} = 0$, suppose that $M_y = N_x$

We will construct a function $\psi(x, y)$ such that $\psi_x = M$ and $\psi_y = N$ to show that this diff. eq. is exact

To ensure that $\psi_x = M$, we can integrate $M(x, y)$ to get:

$$\psi(x, y) = \int M(x, y)dx$$

This will determine $\psi(x, y)$ up to a constant with respect to x.

That is, we can write $\psi(x, y)$ as:

$$\psi(x, y) = Q(x, y) + h(y)$$

To determine the constant with respect to x, which is $h(y)$, we take the partial derivative with respect y to get:

$$\psi_y(x, y) = Q_y(x, y) + h'(y)$$

Using the fact that $\psi_y(x, y) = N(x, y)$ we get that:

$$h'(y) = N(x, y) - Q_y(x, y)$$

We can integrate this, with respect to y to find $h(y)$, giving us the desired $\psi(x, y)$ up to a constant, c.

Note: To do the last integration we need $N(x, y) - Q_y(x, y)$ to be a function of just y. We can see this by showing the partial derivative with respect to x is 0.

Exact Equations - Example

Example: Solve the differential equation:

$$y \cdot sin(x) + e^y + (xe^y - cos(x) + 1)y' = 0$$

Solution: Since this is not linear, nor separable, we need to check if this is an exact equation by comparing M_y to N_x

$M_y = (y \cdot sin(x) + e^y)_y = sin(x) + e^y$

$N_x = (xe^y - cos(x) + 1)_x = e^y + sin(x)$

Since $M_y = N_x$, we can conclude that this diff. eq. is exact.

So, there exists a $\psi(x, y)$ such that $\psi_x = M(x, y) = y \cdot sin(x) + e^y$

We can begin to find $\psi(x, y)$ by integrating this with respect to x to get:

$$\psi(x, y) = \int y \cdot sin(x) + e^y dx = -y \cdot cos(x) + xe^y + h(y)$$

To find $h(y)$, we compute $\psi_y(x, y)$ which we know equals $N(x, y)$:

$$\psi_y(x, y) = (-y \cdot cos(x) + xe^y + h(y))_y = -cos(x) + xe^y + h'(y)$$

Setting this equal to $N(x, y) = xe^y - cos(x) + 1$, we can conclude that $h'(y) = 1$ and thus $h(y) = y$ up to a constant, c.

So, we get the implicit solution: $-y \cdot cos(x) + xe^y + y = c$

Second Order Differential Equations

Thus far, we have studied First-Order Diff. Eq., which have the form:

$$\frac{dy}{dt} = f(t, y)$$

We will now begin our study of Second-Order Differential Equations, which have the form:

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

We will restrict our discussion to **linear** second-order differential equations, which can be written as:

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

We will begin our conversation by looking at differential equations where $G(t) = 0$, which are called **homogeneous** and, furthermore, where the coefficients are constant.

Such Second-Order Homogeneous Differential Equations with Constant Coefficients look like:

$$ay'' + by' + cy = 0$$

We will study Second-Order Nonhomogeneous Differential Equations with Constant Coefficients in coming lectures.

Second Order Differential Equations - Example

Example: Find solutions of the differential equation:

$$y'' - y = 0$$

Can we guess solutions to this differential equation based on our knowledge of derivatives?

Re-writing it as $y'' = y$ may help to figure it out.

The easiest solution to see is $y = e^t$

Another function that works is $y = e^{-t}$

Are there others?

We may notice that constant multiples of these are solutions.

i.e. $y = 3e^t$, $y = 5e^t$, $y = c_1 e^t$, and $y = c_2 e^{-t}$ are all solutions because we can bring the constant outside of derivative.

Similarly due to linearity of derivatives, if y_1 and y_2 are solutions then their sum, $y = y_1 + y_2$, is also a solution.

Combining these two ideas, we can conclude that all functions of the form:

$$y = c_1 e^t + c_2 e^{-t}$$

are solutions.

Second Order Differential Equations - Example

We will use this example to lead us to solutions of diff. eq. of the form:

$$ay'' + by' + cy = 0$$

We will look for solutions of the form $y = e^{rt}$ for some value of r

For $y = e^{rt}$ to be a solution, it must satisfy the differential equation. To test if it does, we will insert $y' = re^{rt}$ and $y'' = r^2e^{rt}$ into the diff. eq.

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

Factoring out e^{rt}, we get that:

$$e^{rt}\left(ar^2 + br + c\right) = 0$$

Since $e^{rt} \neq 0$, we can conclude that for $y = e^{rt}$ to be a solution, then

$$\left(ar^2 + br + c\right) = 0$$

This is very powerful because it allows us to solve our original diff. eq. by solving this quadratic equation, called the **characteristic equation**.

Note: Because of how the characteristic equation is constructed, its coefficients are the same as the coefficients of the original differential equation.

Second Order Differential Equations - Example 2

Example: Find solutions to the differential equation:

$$y'' + 15y' - 34y = 0$$

Solution: To find solutions of the form $y = e^{rt}$, we can solve the characteristic equation:

$$r^2 + 15r - 34 = 0$$

We can compute the solutions by factoring: $r^2 + 15r - 34 = (r + 17)(r - 2)$

which gives the values $r = -17$ and $r = 2$.

So, we can conclude that both $y_1 = e^{2t}$ and $y_2 = e^{-17t}$ are solutions.

We can, further, conclude that for constants c_1, c_2 all functions of the form:

$$y = c_1e^{2t} + c_2e^{-17t}$$

are solutions to the differential equation.

Second Order Differential Equations - Example with IVP

Example: Find a solution to the Initial Value Problem:

$$y'' + y' - 12y = 0 \quad \text{with } y(0) = 2, y'(0) = 1$$

Solution: To solve the IVP, we should first solve the Diff. Eq.
To find solutions $y = e^{rt}$, we can solve the characteristic equation:

$$r^2 + r - 12 = 0$$

We can compute the solutions by factoring: $r^2 + r - 12 = (r+4)(r-3)$

which gives the values $r = -4$ and $r = 3$

So, we can conclude that both $y_1 = e^{-4t}$ and $y_2 = e^{3t}$ are solutions.

We can, further, conclude that for constants c_1, c_2 all functions of the form:

$$y = c_1 e^{-4t} + c_2 e^{3t}$$

are solutions to the differential equation.

To solve the Initial Value Problem, we need to use the initial conditions to find c_1 and c_2

Second Order Differential Equations - Example with IVP

Example: Find solutions to the Initial Value Problem:

$$y'' + y' - 12y = 0 \quad \text{with } y(0) = 2, y'(0) = 1$$

Solution: We can, further, conclude that for constants c_1, c_2 all functions of the form:

$$y = c_1 e^{-4t} + c_2 e^{3t}$$

are solutions to the differential equation.
Imposing the Initial Condition $y(0) = 2$ we get that:

$$2 = c_1 e^{(-4 \cdot 0)} + c_2 e^{(3 \cdot 0)} = c_1 + c_2$$

To use the Initial Condition $y'(0) = 1$ we need the derivative $y' = -4c_1 e^{-4t} + 3c_2 e^{3t}$, which yields:

$$1 = y'(0) = -4c_1 + 3c_2$$

Substituting $c_1 = 2 - c_2$ (from the equation for the first initial condition) into the equation for the second initial condition we get:

$$1 = -4c_1 + 3c_2 = -4(2 - c_2) + 3c_2 = -8 + 7c_2$$

Adding 8 and dividing by 7, we can conclude that $c_2 = \frac{9}{7}$ and $c_1 = 2 - \frac{9}{7} = \frac{5}{7}$
Conclusion: The solution to the initial value problem is:

$$y = \frac{5}{7} e^{-4t} + \frac{9}{7} e^{3t}$$

Existence and Uniqueness Theorem for 2nd-order Diff. Eq.

For linear, first-order differential equations, we saw that if the functions $p(t)$ and $g(t)$ are continuous on the open interval $\alpha < t < \beta$ containing $t = t_o$, then there exists a unique function $y = y_1(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for all $\alpha < t < \beta$ that satisfies an initial condition $y(t_o) = y_o$.
There is a similar Existence and Uniqueness Theorem for solutions to linear, second-order differential equations.
Theorem: Existence and Uniqueness Theorem for Second-Order Linear Diff Eq
If the functions $p(t)$, q(t), and $g(t)$ are continuous on the open interval $\alpha < t < \beta$ containing $t = t_o$, then there exists a unique function $y = y_1(t)$ that satisfies the differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

for all $\alpha < t < \beta$ and satisfies initial conditions $y(t_o) = y_o$ and $y'(t_o) = y'_o$.
This tells us that, on such an interval where p, q, and g are continuous, every initial value problem with a second-order, linear diff. eq. has a solution and that the solution is unique on that interval.
Note: Since such a diff. eq. is second-order, these guaranteed solutions must be twice differentiable.

Existence and Uniqueness Theorem - Example

Example: Find the longest interval on which the Existence and Uniqueness Theorem guarantees a unique, twice differentiable soltution to:

$$t(t+4)y'' + y' + ty = 3 \text{ with } y(1) = 4 \text{ and } y'(1) = -1$$

An IVP of the form $y'' + p(t)y' + q(t)y = g(t)$, with initial conditions at $t = t_o$, has a unique solution on an interval containing $t = t_o$ where $p(t)$, $q(t)$, and $g(t)$ are continuous.

Writing our diff. eq. in this standard form, by dividing by $t(t+4)$, we have:

$$y'' + \frac{1}{t(t+4)}y' + \frac{1}{t+4}y = \frac{3}{t(t+4)}$$

$p(t) = \frac{1}{t(t+4)}$ has discontinuities at $t = 0$ and $t = -4$

$q(t) = \frac{1}{t+4}$ has a discontinuity at $t = -4$

$g(t) = \frac{3}{t(t+4)}$ has discontinuities at $t = 0$ and $t = -4$

Thus, all three are continuous on the intervals $(-\infty, -4)$, $(-4, 0)$, and $(0, \infty)$. Since our initial condition is at $t_o = 1 \in (0, \infty)$, the largest interval on which a solution to the IVP is guaranteed to exist and be unique is the interval $(0, \infty)$. Note: If our initial conditions were at $t_o = -2$ then the largest interval would be $(-4, 0)$, since that is the largest interval containing that initial value of t.

Wronskian - Families of Solutions

We saw that the differential equation:

$$ay'' + by' + cy = 0$$

has an associated **characteristic equation** $a^2r + br + c = 0$ such that the solutions of the differential equation are $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ where the constants r_1, r_2 are solutions to the characteristic equation.

Moreover, we will show that if y_1 and y_2 are solutions to $ay'' + by' + cy = 0$ then there is an infinite family of solutions of the form:

$$y = c_1 y_1(t) + c_2 y_2(t)$$

But are there more solutions to the differential equation that we're missing?

Or, conversely, is the above family of solutions the general solution?

We will show that this is, in fact, the general solution.

The statement of our theorem will be for a more general class of differential equations:

$$y'' + p(t)y' + q(t)y = 0$$

Wronskian - Families of Solutions

We will start by showing that if y_1 and y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$ then $y = c_1y_1(t) + c_2y_2(t)$ is a solution, as well, for any constant values of c_1, c_2

<u>**Proof**</u>: Suppose that y_1 and y_2 are solutions to $\mathbf{y'' + p(t)y' + q(t)y = 0}$

Testing to see if $c_1y_1(t) + c_2y_2(t)$ is a solution to the differential equation, we check to see if:

$$(c_1y_1(t) + c_2y_2(t))'' + p(t)\left(c_1y_1(t) + c_2y_2(t)\right)' + q(t)\left(c_1y_1(t) + c_2y_2(t)\right) = 0$$

Because we have linearity of the derivative, we can split the left hand side up as:

$$(c_1y_1(t) + c_2y_2(t))'' + p(t)\left(c_1y_1(t) + c_2y_2(t)\right)' + q(t)\left(c_1y_1(t) + c_2y_2(t)\right)$$
$$= c_1y_1''(t) + c_2y_2''(t) + p(t)c_1y_1'(t) + p(t)c_2y_2'(t) + q(t)c_1y_1(t) + q(t)c_2y_2(t)$$

Rearranging the right hand side to separate the y_1 and y_2 terms we get:

$$= c_1y_1''(t) + p(t)c_1y_1'(t) + q(t)c_1y_1(t) + c_2y_2''(t) + p(t)c_2y_2'(t) + q(t)c_2y_2(t)$$

We can then factor out c_1 and c_2 from their respective terms to get:

$$= c_1\underbrace{\left(y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)\right)}_{=0} + c_2\underbrace{\left(y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)\right)}_{=0} = 0$$

Since y_1, y_2 are solutions to the **differential equation**, we can see that this reduces to 0. Thus, $y = c_1y_1(t) + c_2y_2(t)$ is a solution as well.

Wronskian - General Solutions

Theorem: Suppose that y_1 and y_2 are solutions to the differential equation:

$$y'' + p(t)y' + q(t)y = 0$$

Let $y(t)$ be a solution which has the initial conditions: $y(t_o) = y_o,\ \ y'(t_o) = y'_o$

Then there are always constants c_1, c_2 so that:

$$y = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$$

satisfies the initial value problem if and only if:

$$y_1 y'_2 - y'_1 y_2 \neq 0 \qquad \text{at } t = t_o$$

We just saw that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution to the Diff. Eq.

For y to be a solution to the IVP we must be able to find c_1, c_2 that solve:

$$c_1 y_1(t_o) + c_2 y_2(t_o) = y_o$$
$$c_1 y'_1(t_o) + c_2 y'_2(t_o) = y'_o$$

From LA there are unique c_1, c_2 if and only if the determinant below is non-zero:

$$W = \begin{vmatrix} y_1(t_o) & y_2(t_o) \\ y'_1(t_o) & y'_2(t_o) \end{vmatrix} = y_1(t_o)y'_2(t_o) - y'_1(t_o)y_2(t_o)$$

This value W is called the **Wronskian determinant**

In other words, if the y_1 and y_2 are solutions to the differential equation:

$$y'' + p(t)y' + q(t)y = 0$$

and $W[y_1, y_2] \neq 0$, then the general solution of the differential equation is:

$$y = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$$

We call solutions y_1 and y_2, with $W[y_1, y_2] \neq 0$, a **fundamental set**.

Wronskian - General Solutions of Exponentials

Example: Suppose that $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are solutions to a differential equation of the form $y'' + p(t)y' + q(t)y = 0$. Check if $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is the general solution.

We can check if this is the general solution by checking to see if the Wronskian, $W \neq 0$.

Since $W = y_1 y_2' - y_1' y_2$, we'll need to know y_1', y_2' as well.

$y_1' = r_1 e^{r_1 t}$ and $y_2' = r_2 e^{r_2 t}$

Now we can compute the Wronskian to be: $W = e^{r_1 t} r_2 e^{r_2 t} - r_1 e^{r_1 t} e^{r_2 t}$

Factoring out $e^{r_1 t} e^{r_2 t}$, we have: $W = e^{r_1 t} e^{r_2 t} (r_2 - r_1)$

Since $e^{r_1 t} e^{r_2 t} \neq 0$, we conclude $W = 0 \Leftrightarrow r_1 = r_2$

Conclusion: If $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are solutions to a differential equation of the form $y'' + p(t)y' + q(t)y = 0$, then $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is the general solution if and only if $r_2 \neq r_1$.

Wronskian - General Solutions of Exponentials Example

Example: Find the general solution of:

$$y'' + y' - 12y = 0$$

To start, we find solutions to the characteristic equation: $r^2 + r - 12 = 0$

This can be factored as: $r^2 + r - 12 = (r - 3)(r + 4)$

Which yields the roots: $r_1 = 3$ and $r_2 = -4$

Conclusion: The general solution to $y'' + y' - 12y = 0$ is:

$$y = c_1 e^{3t} + c_2 e^{-4t}$$

Notice that this is very similar to the problem we did previously. The difference is that, now, we can conclude that this is the general solution rather than just an infinite set of solutions.

Abel's Theorem

We have shown that if the y_1 and y_2 are solutions to the differential equation:

$$y'' + p(t)y' + q(t)y = 0$$

and $W[y_1, y_2](t_o) \neq 0$, then the general solution of the differential equation is:

$$y = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$$

This relies on evaluating $W[y_1, y_2]$ at a value $t = t_o$.
A question to consider is: Does it matter which value $t = t_o$ we choose?
For homogeneous, second-order diff. eq. with constant coefficients of the form $ay'' + by' + cy = 0$, with r_1, r_2 for solutions to the characteristic polynomial, we found that the condition only depends on r_1, r_2.
In particular, we found: that $W[e^{r_1 t}, e^{r_2 t}] \neq 0 \Leftrightarrow r_1 \neq r_2$
So, it did not depend on which value $t = t_o$ at which W was evaluated.
Abel's Theorem shows that whether or not $W[y_1, y_2] = 0$ is independent of t_o for linear, homogeneous, second-order differential equations.
Theorem: Abel's Theorem
If y_1 and y_2 are solutions to the diff. eq.:

$$y'' + p(t)y' + q(t)y = 0$$

with $p(t)$ and $q(t)$ continuous on an open interval (α, β), then:

$$W[y_1, y_2](t) = c \cdot e^{\left(-\int p(t)dt\right)}$$

where c is a constant depending on y_1, y_2 but independent of t.
Since $e^{power} \neq 0$, we see that $W[y_1, y_2] = 0 \Leftrightarrow c = 0$, which is independent of t

Abel's Theorem

Theorem: Abel's Theorem
If y_1 and y_2 are solutions to the diff. eq.:

$$y'' + p(t)y' + q(t)y = 0$$

with $p(t)$ and $q(t)$ continuous on an open interval (α, β), then:

$$W[y_1, y_2](t) = c \cdot e^{\left(-\int p(t)dt\right)}$$

where c is a constant depending on y_1, y_2 but independent of t.
Proof: Since y_1, y_2 are solutions to the differential equation, we know that:

$$y_1'' + p(t)y_1' + q(t)y_1 = 0$$
$$y_2'' + p(t)y_2' + q(t)y_2 = 0$$

We can eliminate the last term by multiplying the top equation by y_2 and the bottom equation by y_1 and subtracting the top from the bottom to get:

$$y_2''y_1 - y_1''y_2 + p(t)(y_2'y_1 - y_1'y_2) = 0$$

We can write this equation in terms of $W = W[y_1, y_2] = y_1y_2' - y_2y_1'$ by computing its derivative:

$$\begin{aligned} W' &= (\underbrace{y_1y_2'}\, \underbrace{-y_2y_1'})' \\ &= \underbrace{\cancel{y_1'y_2'} + y_1y_2''}\, \underbrace{-\cancel{y_2'y_1'} - y_2y_1''} = y_1y_2'' - y_2y_1'' \end{aligned}$$

So, our equation becomes a first-order linear diff. eq. of W: $W' + p(t)W = 0$
Rearranging this equation as: $W' = -p(t)W$

Solving for W as a separable diff. eq. gives the condition: $W = c \cdot e^{-\left(\int p(t)dt\right)}$

Homogeneous Diff. Eq. w/ Const. Coeff - Complex Case

We saw that the differential equation:

$$ay'' + by' + cy = 0$$

has an associated **characteristic equation** $ar^2 + br + c = 0$ such that the solutions of the differential equation are $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ where the constants r_1, r_2 are solutions to the characteristic equation.

Further, we saw that if $r_1 \neq r_2$, we could build the general solution:

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

However, since the characteristic equation is a quadratic polynomial, there are three possibilities for the roots.

The case above is when the char. poly. has 2 distinct, real roots.

Another possibility is that the char. poly. has 2 complex roots

And the final possibility is that the char. poly. has 1 repeated, real root.

We will study these last two possibilities, starting with the complex case.

Homogeneous Diff. Eq. w/ Const. Coeff - Complex Case

Theorem: If $y = u(t) + iv(t)$ is a complex-valued solution of a differential equation of the form:

$$y'' + p(t)y' + q(t)y = 0$$

then $u(t)$ and $v(t)$ are both real-valued solutions to this differential equation.

Proof: Since $u(t) + iv(t)$ is a solution to the above diff. eq., we know that:

$$\begin{aligned} 0 &= (u(t) + iv(t))'' + p(t)\,(u(t) + iv(t))' + q(t)\,(u(t) + iv(t)) \\ &= (u''(t) + p(t)u'(t) + q(t)u(t)) + i(v''(t) + p(t)v'(t) + q(t)v(t)) \end{aligned}$$

Where we have separated the terms into real and imaginary parts.

A complex number is zero if and only if its real and imaginary parts are both 0.

Since the complex-valued function above equals zero, we can conclude that the real and imaginary parts are both 0.

That is: $\quad (u''(t) + p(t)u'(t) + q(t)u(t)) = 0$

$$(v''(t) + p(t)v'(t) + q(t)v(t)) = 0$$

But this means that both the real and imaginary parts of the complex-valued solution are solutions themselves.

Homogeneous Diff. Eq. w/ Const. Coeff - Complex Case

Theorem: If $y = u(t) + iv(t)$ is a complex-valued solution of a differential equation of the form:

$$y'' + p(t)y' + q(t)y = 0$$

then $u(t)$ and $v(t)$ are both real-valued solutions to this differential equation.

We have shown that, since $u(t), v(t)$ are solutions:

$$y = c_1 u(t) + c_2 v(t)$$

is an infinite family of solutions.

Is it the general solution?

We will have to check if the Wronskian, $W \neq 0$ to answer that.

Before we do that, let's study the solutions $u(t)$ and $v(t)$

Homogeneous Diff. Eq. w/ Const. Coeff - Complex Case

Suppose that the differential equation $ay'' + by' + cy = 0$ has a characteristic equation that yields complex roots, r_1, r_2.

That is, the solutions to $ar^2 + br + c = 0$ are complex.

Since this the a real valued polynomial, the complex solutions must come in conjugate pairs, $\gamma \pm i\mu$, where γ and μ are real.

$r_1 = \gamma + i\mu$ gives us a complex solution to the differential equation, since r_1 is a solution to the characteristic equation.

More specifically, $y = e^{(\gamma+i\mu)t}$ is a complex-valued solution.

We just saw that, if we can split y into its real and imaginary parts, then each will be a real-valued solution to the diff. eq.

How do we split $y = e^{(\gamma+i\mu)t}$ into its real and imaginary parts?

Homogeneous Diff. Eq. w/ Const. Coeff - Complex Case

How do we split $y = e^{(\gamma+i\mu)t}$ into its real and imaginary parts?
The first step is to use our rules of exponents to rewrite y as:

$$y = e^{(\gamma+i\mu)t} = e^{\gamma t} \cdot e^{i\mu t}$$

Writing it this way, if we can split $e^{i\mu t}$ into real and imaginary parts then we will have y split into real and imaginary parts.
We can use Euler's Formula to write:

$$e^{i\mu t} = cos(\mu t) + i \cdot sin(\mu t)$$

Which gives us:

$$\begin{aligned} y =& e^{\gamma t} \cdot e^{i\mu t} \\ =& e^{\gamma t} \cdot (cos(\mu t) + i \cdot sin(\mu t)) \\ =& e^{\gamma t} \cdot cos(\mu t) + i \cdot e^{\gamma t} \cdot sin(\mu t) \end{aligned}$$

Thus, we've written y in its real and imaginary parts:

$$y = e^{\gamma t} \cdot cos(\mu t) + i \cdot e^{\gamma t} \cdot sin(\mu t)$$

Conclusion: $y_1 = e^{\gamma t} \cdot cos(\mu t)$ and $y_2 = e^{\gamma t} \cdot sin(\mu t)$ are real solutions to the differential equation:

$$ay'' + by' + cy = 0$$

where $r = \gamma + i \cdot \mu$ is a complex solution to the char. eq. $ar^2 + br + c = 0$

Homogeneous Diff. Eq. w/ Const. Coeff - Complex Case

We just saw that if $r = \gamma + i \cdot \mu$ is a complex solution to its char. eq. then

$$y_1 = e^{\gamma t} \cdot cos(\mu t) \text{ and } y_2 = e^{\gamma t} \cdot sin(\mu t)$$

are real solutions to the differential equation:

$$ay'' + by' + cy = 0$$

We have also seen that if y_1 and y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$ then $y = c_1 y_1(t) + c_2 y_2(t)$ is a solution, as well, for any constant values of c_1, c_2

Combining these ideas, we see that, for any constants c_1, c_2:

$$y = c_1 e^{\gamma t} \cdot cos(\mu t) + c_2 e^{\gamma t} \cdot sin(\mu t) \text{ is a solution.}$$

To conclude that this is the General Solution, we need the Wronskian, $W \neq 0$

Computing the W, we can show that $W = \mu e^{(2\gamma t)}$ and, thus:

$$W = 0 \text{ if and only if } \mu = 0$$

But, in the case that $\mu = 0$, the roots to the char. eq. are not complex.

Conclusion: If $r = \gamma + i \cdot \mu$ is a complex solution to its char. eq. then

$$y = c_1 e^{\gamma t} \cdot cos(\mu t) + c_2 e^{\gamma t} \cdot sin(\mu t)$$

is the General Solution to the differential equation: $ay'' + by' + cy = 0$

Homogeneous Diff. Eq. w/ Const. Coeff - Complex Case

Example: Find the General Solution of:

$$y'' + 2y' + 5y = 0$$

To start, we find solutions to the characteristic equation: $r^2 + 2r + 5 = 0$

This can't be factored, so we can use the quadratic formula to find the roots:

$$r_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

Using the root $r = -1 + 2i$, we have that:

$$y = c_1 e^{-1t} \cdot cos(2t) + c_2 e^{-1t} \cdot sin(2t)$$

is the General Solution.

Homogeneous Diff. Eq. w/ Const. Coeff - Complex Case

Example: Find the Solution of the IVP:

$$y'' - 4y' + 13y = 0 \qquad \text{with } y(0) = 1 \text{ and } y'(0) = 8$$

To start, we find the gen. sol., by solving the char. eq.: $r^2 - 4r + 13 = 0$
This can't be factored, so we can use the quadratic formula to find the roots:

$$r_{1,2} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 13}}{2 \cdot 1} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

Using the root $r = 2 + 3i$, we have that the General Solution is:

$$y = c_1 e^{2t} \cdot cos(3t) + c_2 e^{2t} \cdot sin(3t)$$

Now that we have the Gen. Sol., we need to impose the Initial Conditions.

$$1 = y(0) = c_1 \underbrace{e^{2 \cdot 0}}_{=1} \underbrace{cos(3 \cdot 0)}_{=1} + c_2 \underbrace{e^{3 \cdot 0}}_{=1} \underbrace{sin(3 \cdot 0)}_{=0} = c_1 \qquad \Leftrightarrow c_1 = 1$$

To use $y'(0) = 8$, we need to compute the derivative using the product rule:

$$y' = 2c_1 e^{2t} cos(3t) - 3c_1 e^{2t} sin(3t) + 2c_2 e^{2t} sin(3t) + 3c_2 e^{2t} cos(3t)$$

Evaluating $y'(0)$, we get: $8 = y'(0) = 2c_1 + 3c_2 = 2 + 3c_2 \qquad \Leftrightarrow c_2 = 2$

Conclusion: the solution to the IVP is: $y = e^{2t} \cdot cos(3t) + 2e^{2t} \cdot sin(3t)$

Homogeneous Diff. Eq. w/ Const. Coeff - Repeated Roots

We saw that if the number r is a solution to the char. eq.: $ar^2 + br + c = 0$ then the function $y = e^{rt}$ is a solution to the diff. eq.: $ay'' + by' + cy = 0$

We, also, saw that if $y_1(t)$ and $y_2(t)$ are solutions to the diff. eq.: $ay'' + by' + cy = 0$ and the Wronskian $W[y_1, y_2] \neq 0$ then the general solution to the diff. eq. is:

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

Whether r was real or complex, we could always find two functions y_1, y_2 to form the general solution.

There is one more case that can come up.

What if characteristic equation only has one, repeated, real root?

That is, what if $ar^2 + br + c = a(r - \gamma)^2$ for some $\gamma \in \mathbf{R}$?

We know that $y_1 = e^{\gamma t}$ is a solution. But can we find a second solution y_2 to build the general solution?

We saw for complex roots that the solutions looked like $e^{\gamma t} cos(\mu t), e^{\gamma t} sin(\mu t)$

This guides us to check for the second solution as a function of the form $v(t)e^{\gamma t}$ for some function $v(t)$.

Homogeneous Diff. Eq. w/ Const. Coeff - Repeated Roots

We saw that if $y_1(t)$ and $y_2(t)$ are solutions to the diff. eq.: $ay'' + by' + cy = 0$ and the Wronskian $W[y_1, y_2] \neq 0$ then the general solution to the diff. eq. is:

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

If the characteristic equation $ar^2 + br + c = a(r - \gamma)^2$ for some $\gamma \in \mathbf{R}$, we know that $y_1 = e^{\gamma t}$ is a solution.

Our work with complex guides us to think that the second solution may be of the form:

$$y_2 = v(t)e^{\gamma t}$$

We can show that for $y_2 = v(t)e^{\gamma t}$ to be a solution, then $v(t) = t$

In other words, if the characteristic equation has a repeated root, γ, then two solutions to the differential equation are:

$$y_1 = e^{\gamma t} \qquad \text{and} \qquad y_2 = te^{\gamma t}$$

Furthermore, we can show that the Wronskian, $W[y_1, y_2] \neq 0$.

So, we can conclude:

If the characteristic equation $ar^2 + br + c = a(r - \gamma)^2$ for some $\gamma \in \mathbf{R}$, then the general solution to the differential equation $ay'' + by' + cy = 0$ is:

$$y(t) = c_1 e^{\gamma t} + c_2 te^{\gamma t}$$

Hom. Diff. Eq. Const. Coeff - Repeated Roots $v(t)$

Suppose that the differential equation $ay'' + by' + cy = 0$, with characteristic equation $ar^2 + br + c = 0$ has a repeated real root, γ.

We want to find a function $v(t)$ so that $y = v(t)e^{\gamma t}$ is a solution.

To begin, let's reduce the equation by dividing by a and use $B = \frac{b}{a}$ and $C = \frac{c}{a}$:

Diff. Eq: $y'' + By' + Cy = 0$ and Char. Eq.: $r^2 + Br + C = 0$

Since the char. eq. has a repeated root, γ, it can be written as:

$$r^2 - 2\gamma r + \gamma^2 = (r - \gamma)(r - \gamma) = 0$$

Taking the first and second derivative of $y = v(t)e^{\gamma t}$ we get:

$$y' = v'(t)e^{\gamma t} + \gamma v(t)e^{\gamma t} \quad , \quad y'' = v''(t)e^{\gamma t} + 2\gamma v'(t)e^{\gamma t} + \gamma^2 v(t)e^{\gamma t}$$

Putting y, y', and y'' into our differential equation, we get:

$$v''(t)e^{\gamma t} + \cancel{2\gamma v'(t)e^{\gamma t}} + \cancel{\gamma^2 v(t)e^{\gamma t}} - \cancel{2\gamma}(\cancel{v'(t)e^{\gamma t}} + \cancel{\gamma v(t)e^{\gamma t}}) + \cancel{\gamma^2 v(t)e^{\gamma t}} = 0$$

So, we need $v''(t)e^{\gamma t} = 0$, and thus $v''(t) = 0$ since $e^{\gamma t} \neq 0$

We can conclude that $y = v(t)e^{\gamma t}$ is a solution to the diff. eq. if $v''(t) = 0$

Since $v(t) = t$ satisfies $v''(t) = 0$, we can conclude that"'

$$y = te^{\gamma t} \text{ is a solution}$$

Hom. Diff. Eq. Const. Coeff - Repeated Roots Wronskian

For the solutions $y_1 = e^{\gamma t}$ and $y_2 = te^{\gamma t}$, we wish to show that $W[y_1, y_2] \neq 0$

To do this, we wish to compute the derivatives y_1' and y_2'

$y_1' = \gamma e^{\gamma t}$

$y_2' = e^{\gamma t} + \gamma te^{\gamma t}$

Using these to calculate the Wronskian, we get:

$$W[y_1, y_2] = y_1 \cdot y_2' - y_1' \cdot y_2 = e^{\gamma t} \cdot (e^{\gamma t} + \gamma te^{\gamma t}) - \gamma e^{\gamma t} \cdot te^{\gamma t}$$

Distributing $e^{\gamma t}$:

$$= e^{\gamma t} \cdot e^{\gamma t} + \cancel{e^{\gamma t} \cdot \gamma te^{\gamma t} - \gamma e^{\gamma t} \cdot te^{\gamma t}} = e^{\gamma t} \cdot e^{\gamma t}$$

Since $e^{\gamma t} \neq 0$, we can conclude that $W[y_1, y_2] \neq 0$

Hom. Diff. Eq. w/ Const. Coeff - Repeated Roots Example

Example: Find the solution to the Initial Value Problem:

$$y'' + 8y' + 16y = 0 \quad \text{with } y(0) = 5 \text{ and } y'(0) = -3$$

To solve an initial value problem, we first find the general solution.
We start to find the gen. sol. by looking at the characteristic equation:

$$r^2 + 8r + 16 = 0$$

We can factor this to find the repeated root $r = -4$
So, the general solution is: $y(t) = c_1 e^{-4t} + c_2 t e^{-4t}$
To use the second initial condition we need to compute $y'(t)$:

$$y'(t) = -4c_1 e^{-4t} + c_2 e^{-4t} + (-4)c_2 t e^{-4t}$$

Using the initial condition $y(0) = 5$ we get that:

$$5 = y(0) = c_1 e^{-4\cdot 0} + c_2 \cdot 0 e^{-4\cdot 0} = c_1$$

Using the initial condition $y'(0) = -3$ we get that:

$$-3 = y'(0) = -4c_1 e^{-4\cdot 0} + c_2 e^{-4\cdot 0} + (-4)c_2 \cdot 0 \cdot e^{-4\cdot 0} = -4c_1 + c_2$$

Since $c_1 = 5$ we can solve for c_2 to get: $c_2 = 17$
Thus, the solution to the initial value problem is:

$$y(t) = 5e^{-4t} + 17te^{-4t}$$

Nonhomogeneous Diff. Eq. w/ Const. Coeff

We have now seen how to solve differential equations of the form:

$$ay'' + by' + cy = 0$$

We will now study the non-homogeneous case:

$$ay'' + by' + cy = g(t)$$

As we did with the homogeneous case, we will build much of our theory for more general linear differential equations:

$$y'' + p(t)y' + q(t)y = g(t)$$

We will start by showing that:

Theorem: If Y_1 and Y_2 are two solutions to the nonhomogeneous diff. eq.:

$$y'' + p(t)y' + q(t)y = g(t)$$

then $Y_1 - Y_2$ is a solution to the associated homogeneous diff. eq.:

$$y'' + p(t)y' + q(t)y = 0$$

Proof: Testing $Y_1 - Y_2$ in the homogeneous diff. eq. we get:

$(Y_1 - Y_2)'' + p(t)(Y_1 - Y_2)' + q(t)(Y_1 - Y_2) =$

Since Y_1 and Y_2 are solutions to the nonhomogeneous diff. eq.

Thus, we can conclude that $Y_1 - Y_2$ is a solution to the homog. diff. eq.

Nonhomogeneous Diff. Eq. w/ Const. Coeff

Theorem: If Y_1 and Y_2 are two solutions to the nonhomogeneous diff. eq.:

$$y'' + p(t)y' + q(t)y = g(t)$$

then $Y_1 - Y_2$ is a solution to the associated homogeneous diff. eq.:

$$y'' + p(t)y' + q(t)y = 0$$

This theorem allows us to build a general solution to the non-homog. diff. eq.
Recall: that if y_1, y_2 are solutions to the homog. diff. eq. with $W[y_1, y_2] \neq 0$ then the general solution of the homog. diff. eq. can be written as:

$$y_h(t) = c_1 y_1 + c_2 y_2$$

If we can find a particular solutions, $y_p(t)$ to the non-homog. diff. eq., then all solutions $y(t)$ to the non-homog. diff. eq. satisfy:

$$y(t) - y_p(t) = c_1 y_1 + c_2 y_2$$

Rearranging this to solve for $y(t)$ we get the following theorem:
Theorem: If y_p is a particular solution of the non-homogeneous diff. eq.

$$y'' + p(t)y' + q(t)y = g(t)$$

then the general solution can be written in terms of y_1, y_2, solutions to the associated homog. diff. eq. (with $W[y_1, y_2] \neq 0$) and constants c_1, c_2 as:

$$y(t) = y_p(t) + c_1 y_1 + c_2 y_2$$

Nonhomogeneous DE w/ Const. Coeff - Particular Solutions

Theorem: If y_p is a particular solution of the non-homogeneous diff. eq.

$$y'' + p(t)y' + q(t)y = g(t)$$

then the general solution can be written in terms of y_1, y_2, solutions to the associated homog. diff. eq. (with $W[y_1, y_2] \neq 0$) and constants c_1, c_2 as:

$$y(t) = y_p(t) + c_1 y_1 + c_2 y_2$$

Since we learned how to find the general solution to any homogeneous second-order differential equation with constant coefficients, we can solve any nonhomogeneous second-order differential equation with constant coefficients of the form:

$$ay'' + by' + cy = g(t)$$

so long as we can find one particular solution $y_p(t)$.

We now need a technique to find one particular solution $y_p(t)$ to the non-homogeneous differential equation.

The particular solution, $y_p(t)$, will depend on $g(t)$

Thus, our approach will use $g(t)$ to determine the form of $y_p(t)$.

We will look at several examples where we find $y_p(t)$.

Nonhomogeneous DE w/ Const. Coeff - Particular Solutions

Example: Find a particular solution to:

$$y'' - y' - 6y = e^{2t}$$

We need to try to find a function $y(t)$ that is a solution to this diff. eq.

That is, we need a function $y(t)$ that balances both sides of the equation

Since e^{2t} appears on the right hand side, we will need e^{2t} to be on the left hand side in order for both sides to be equal.

The easiest way to achieve this is try the form $y(t) = Ae^{2t}$ for some A

To see if $y(t)$ is a solution, for some A, we check $y(t)$ in the diff. eq.

To do this, we need to compute: $y'(t) = 2Ae^{2t}$ and $y''(t) = 4Ae^{2t}$

Checking this in the differential equation, we get:

$$4Ae^{2t} - 2Ae^{2t} - 6Ae^{2t} = e^{2t}$$

Simplifying the left hand side yields the equation: $-4Ae^{2t} = e^{2t}$

Thus, we can conclude that for $A = -\frac{1}{4}$, $y(t) = Ae^{2t}$ is a solution.

That is, $y(t) = -\frac{1}{4}e^{2t}$ is a particular solution to the diff. eq.

Nonhomogeneous DE w/ Const. Coeff - Particular Solutions

Example: Find the general solution to:

$$y'' - y' - 6y = e^{2t}$$

We saw that if y_p is a particular solution of the non-homogeneous diff. eq.

$$y'' + p(t)y' + q(t)y = g(t)$$

then the general solution can be written in terms of y_1, y_2, solutions to the associated homog. diff. eq. (with $W[y_1, y_2] \neq 0$) and constants c_1, c_2 as:

$$y(t) = y_p(t) + c_1 y_1 + c_2 y_2$$

And we just found a particular solution: $y(t) = -\frac{1}{4}e^{2t}$

Thus, we can find the general solution by finding the solution to the associated homogeneous diff. eq.:

$$y'' - y' - 6y = 0$$

We can find the gen. sol. to the homog. diff. eq. by looking at its char. eq.

$$r^2 - r - 6 = 0$$

Solving this, we get $r_{1,2} = -2, 3$

Thus, the general solution of the homog. diff. eq. is: $y_h = c_1 e^{-2t} + c_2 e^{3t}$

With a particular solution of the non-homog. diff. eq. and the gen. sol. of the assoc. homog. diff. eq. we can build the gen. sol. of the non-homog. diff. eq.:

$$y(t) = -\frac{1}{4}e^{2t} + c_1 e^{-2t} + c_2 e^{3t}$$

Nonhomogeneous Diff. Eq. w/ Const. Coeff IVP

Example: Solve the Initial Value Problem:

$$y'' - 3y' - 18y = e^{-2t} \qquad y(0) = 3 \text{ and } y'(0) = 5$$

Solution: We start by finding the gen. sol. to the associated homogeneous DE:

$$y'' - 3y' - 18y = 0$$

By factoring the char. eq.: $0 = r^2 - 3r - 18 = (r-6)\cdot(r+3)$

We get roots $r_1 = 6$ and $r_2 = -3$ which gives the gen. sol. to the homog. DE:

$$y_h = c_1 e^{6t} + c_2 e^{-3t} \qquad c_1, c_2 \text{ - constant}$$

Next, we need to find a particular solution to the non-homogeneous DE

Based on the RHS e^{-2t}, our particular solution will have the form: $y_p = Ae^{-2t}$

Using $y_p' = -2Ae^{-2t}$ and $y_p'' = 4Ae^{-2t}$ in our DE, we can solve for A:

$$4Ae^{-2t} - 3\cdot\left(-2Ae^{-2t}\right) - 18\cdot Ae^{-2t} = e^{-2t} \Rightarrow -8A = 1 \Rightarrow A = \tfrac{-1}{8}$$

Thus $y_p = \frac{-1}{8}e^{-2t}$, and the gen. sol. of the Non-homog. DE is:

$$y(t) = \tfrac{-1}{8}e^{-2t} + c_1 e^{6t} + c_2 e^{-3t}$$

Now that we have the general solution of the non-homog. DE, we can impose the initial conditions to solve for c_1 and c_2

Doing so, we find that $c_1 = \frac{113}{72}$ and $c_2 = \frac{14}{9}$

Putting this together, we get that the solution to the initial value problem is:

$$y(t) = \tfrac{-1}{8}e^{-2t} + \tfrac{113}{72}e^{6t} + \tfrac{14}{9}e^{-3t}$$

Nonhomogeneous Diff. Eq. Initial Conditions

We found the general solution of the differential equation in the IVP:

$$y'' - 3y' - 18y = e^{-2t} \qquad y(0) = 3 \text{ and } y'(0) = 5$$

to be:

$$y(t) = \frac{-1}{8}e^{-2t} + c_1 e^{6t} + c_2 e^{-3t}$$

Now that we have the general solution of the non-homog. DE, we can impose the initial conditions to solve for c_1 and c_2

Using that $y(0) = 3$ we have:

$$\begin{aligned} 3 = y(0) &= \frac{-1}{8}e^{-2\cdot 0} + c_1 e^{6\cdot 0} + c_2 e^{-3\cdot 0} \\ &= \frac{-1}{8} + c_1 + c_2 \end{aligned}$$

To use the initial condition, $y'(0) = 5$, we need to compute the derivative $y'(t)$

$$y'(t) = \frac{-1}{8} \cdot (-2)\, e^{-2t} + 6 \cdot c_1 e^{6t} - 3 \cdot c_2 e^{-3t}$$

Using that $y'(0) = 5$ we have:

$$\begin{aligned} 5 = y'(0) &= \frac{-1}{8} \cdot (-2)\, e^{-2\cdot 0} + 6c_1 e^{6\cdot 0} - 3c_2 e^{-3\cdot 0} \\ &= \frac{1}{4} + 6 \cdot c_1 - 3 \cdot c_2 \end{aligned}$$

Solving for c_1 in the equation $3 = \frac{-1}{8} + c_1 + c_2$ gives: $c_1 = 3 - \frac{-1}{8} - c_2$

Substituting this c_1 in the equation $5 = \frac{1}{4} + 6 \cdot c_1 - 3 \cdot c_2$ gives:

$$5 = \frac{1}{4} + 6 \cdot \left(3 - \frac{-1}{8} - c_2\right) - 3 \cdot c_2$$

Solving for c_2 we get: $c_2 = \frac{113}{72}$

And this yields $c_1 = \frac{14}{9}$

Nonhomogeneous DE w/ Const. Coeff - Example 1

Example: Find the general solution to:

$$y'' - y' - 6y = \sin(2t)$$

The associated homogeneous diff eq is: $y'' - y' - 6y = 0$
Which has characteristic equation: $r^2 - r - 6 = 0$; Giving the roots $r = -2, 3$
So, the general solution of the assoc. homog. is: $y_h = c_1 e^{-2t} + c_2 e^{3t}$
To solve the nonhomogeneous diff. eq. we'll need a particular solution.
$y_p(t) = A\sin(2t)$ is not enough because $y_p' = 2A\cos(2t)$ cannot be balanced in the differential equation since no other term contains $\cos(2t)$.
So, we need to use: $y_p(t) = A\sin(2t) + B\cos(2t)$
Taking the first and second derivative to use in the diff. eq. we get:
$y_p' = 2A\cos(2t) - 2B\sin(2t)$
$y_p'' = -4A\sin(2t) - 4B\cos(2t)$
Using these in our differential equation gives:

$$-4A\sin(2t) - 4B\cos(2t) - (2A\cos(2t) - 2B\sin(2t)) - 6(A\sin(2t) + B\cos(2t)) = \sin(2t)$$

Comparing coefficients, we get: $-4A + 2B - 6A = 1$ and $-4B - 2A - 6B = 0$

Which yields: $A = \frac{-5}{52}$ and $B = \frac{1}{52}$ thus $y_p = \frac{-5}{52}\sin(2t) + \frac{1}{52}\cos(2t)$
General Solution: $y(t) = \frac{-5}{52}\sin(2t) + \frac{1}{52}\cos(2t) + c_1 e^{-2t} + c_2 e^{3t}$

Nonhomogeneous DE w/ Const. Coeff - Polynomial Ex

Example: Find the general solution to:

$$y'' - y' - 6y = 3t$$

The associated homogeneous diff eq is: $y'' - y' - 6y = 0$
Which has characteristic equation: $r^2 - r - 6 = 0$; Giving the roots $r = -2, 3$
So, the general solution of the assoc. homog. is: $y_h = c_1 e^{-2t} + c_2 e^{3t}$
To solve the nonhomogeneous diff. eq. we'll need a particular solution.
Since $g(t) = 3t$ is a linear polynomial, we'll use a linear polynomial for y_p

$$y_p(t) = At + B$$

Note: even though $g(t) = 3t$ has a constant of 0, we need to choose y_p to be a generic linear polynomial, and leave the constant unknown as B
Taking the first and second derivative to use in the diff. eq. we get:
$y_p' = A$
$y_p'' = 0$
Using these in our differential equation gives:
$0 - (A) - 6(\underbrace{At} + B) = \underbrace{3t} + 0$

Comparing coefficients, we get: $-A - 6B = 0$ and $\underbrace{-6A = 3}$

Which yields: $A = \frac{-1}{2}$ and $B = -\frac{A}{6} = \frac{1}{12}$ thus $y_p = \frac{-1}{2}t + \frac{1}{12}$
General Solution: $y(t) = \frac{-1}{2}t + \frac{1}{12} + c_1 e^{-2t} + c_2 e^{3t}$

Nonhomogeneous DE w/ Const. Coeff - Example 2

Example: Find the general solution to:

$$y'' - y' - 6y = e^{3t}$$

The associated homogeneous diff eq is: $y'' - y' - 6y = 0$
Which has characteristic equation: $r^2 - r - 6 = 0$; Giving the roots $r = -2, 3$
So, the general solution of the assoc. homog. is: $y_h = c_1 e^{-2t} + c_2 e^{3t}$
To solve the nonhomogeneous diff. eq. we'll need a particular solution.
Notice: $y_p = Ae^{3t}$ cannot be a solution to the nonhomogeneous diff. eq. since it is a solution to the assoc. homog. diff. eq. and thus $y_p'' - y_p' - 6y_p = 0 \neq e^{3t}$
Similar to the case of the repeated real root, we need to use: $y_p(t) = Ate^{3t}$
Taking the first and second derivative to use in the diff. eq. we get:

$y_p' = Ae^{3t} + 3Ate^{3t}$

$y_p'' = 3Ae^{3t} + 3Ae^{3t} + 9Ate^{3t} = 6Ae^{3t} + 9Ate^{3t}$

Using these in our differential equation gives:

$$\underbrace{6Ae^{3t}} + \underbrace{9Ate^{3t}} - (\underbrace{Ae^{3t}} + \underbrace{3Ate^{3t}}) + -6\underbrace{Ate^{3t}} = \underbrace{e^{3t}}$$

Comparing coefficients, we get: $\underbrace{6A - A = 1}$ and $\underbrace{9A - 3A - 6A = 0} \to 0$

Which yields: $A = \frac{1}{5}$ thus $y_p = \frac{1}{5}te^{3t}$
General Solution: $y(t) = \frac{1}{5}te^{3t} + c_1 e^{-2t} + c_2 e^{3t}$

Nonhomogeneous DE w/ Const. Coeff - Example 3

Example: Find the general solution to:

$$y'' - 4y' + 4y = e^{2t}$$

The associated homogeneous diff eq is: $y'' - 4y' + 4y = 0$
Which has char eq: $r^2 - 4r + 4 = 0$; Giving the repeated root $r = 2$
So, the general solution of the assoc. homog. is: $y_h = c_1 e^{2t} + c_2 te^{2t}$
To solve the nonhomogeneous diff. eq. we'll need a particular solution.
Notice: $y_p = Ae^{2t}$ cannot be a solution to the nonhomog. diff. eq. since it is a solution to the associated homog. diff. eq. and thus $y_p'' - 4y_p' + 4y_p = 0 \neq e^{2t}$
Similarly $y_p = Ate^{2t}$ is not a solution since it is a sol. to the homog. diff. eq.
We need to extend that trick and use: $y_p(t) = At^2 e^{2t}$
Taking the first and second derivative to use in the diff. eq. we get:
$y_p' = 2Ate^{2t} + 2At^2 e^{2t}$
$y_p'' = 2Ae^{2t} + 4Ate^{2t} + 4Ate^{2t} + 4At^2 e^{2t} = 4At^2 e^{2t} + 8Ate^{2t} + 2Ae^{2t}$
Using these in our differential equation gives:
$\cancel{4At^2 e^{2t}} + \cancel{8Ate^{2t}} + 2Ae^{2t} - 4(\cancel{2Ate^{2t}} + \cancel{2At^2 e^{2t}}) + \cancel{4At^2 e^{2t}} = e^{2t}$
The terms including $t^2 e^{2t}$ and te^{2t} sum to zero.
Comparing coefficients of e^{2t}, we get: $2A = 1$
Which yields: $A = \frac{1}{2}$ thus $y_p = \frac{1}{2} t^2 e^{2t}$
General Solution: $y(t) = \frac{1}{2} t^2 e^{2t} + c_1 e^{2t} + c_2 te^{2t}$

Nonhomogeneous DE w/ Const. Coeff - Example 4

In each of our examples thus far, solving nonhomog. diff. eq.:

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)$$

we have had only one term for $g(t)$

What if $g(t)$ has multiple terms, with the form:

$$y''(t) + p(t)y'(t) + q(t)y(t) = g_1(t) + g_2(t)$$

This does not impact the general solution to the assoc. homog. diff. eq.

But how can we find a particular solution?

Theorem: If y_p and y_q are solutions to:

$y''(t) + p(t)y'(t) + q(t)y(t) = g_1(t)$ and $y''(t) + p(t)y'(t) + q(t)y(t) = g_2(t)$ respectively, then $y_p + y_q$ is a solution of:

$$y''(t) + p(t)y'(t) + q(t)y(t) = g_1(t) + g_2(t)$$

Thus, if we find a particular solution for each term of $g(t)$ separately, then the sum of those solutions is a particular solution to:

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)$$

The proof of this follows immediately from the linearity of derivatives:

$$\begin{aligned}
&(y_p + y_q)'' + p(t)\,(y_p + y_q)' + q(t)\,(y_p + y_q) \\
&\qquad = (y_p)'' + p(t)\,(y_p)' + q(t)\,(y_p) + (y_q)'' + p(t)\,(y_q)' + q(t)\,(y_q) \\
&\qquad = g_1(t) + g_2(t)
\end{aligned}$$

Nonhomogeneous DE w/ Const. Coeff - Example 4

Example: Find the general solution to:

$$y'' - y' - 6y = e^{2t} + e^{3t}$$

The associated homogeneous diff eq is: $y'' - y' - 6y = 0$

Which has characteristic equation: $r^2 - r - 6 = 0$; Giving the roots $r = -2, 3$

So, the general solution of the assoc. homog. is: $y_h = c_1 e^{-2t} + c_2 e^{3t}$

To solve the nonhomogeneous diff. eq. we'll need a particular solution.

We found that $y_p = -\frac{1}{4}e^{2t}$ is a sol. to: $y'' - y' - 6y = e^{2t}$

And that $y_q = \frac{1}{5}te^{3t}$ is a sol. to: $y'' - y' - 6y = e^{3t}$

Thus, $y_p + y_q = -\frac{1}{4}e^{2t} + \frac{1}{5}te^{3t}$ is a particular solution to:

$$y'' - y' - 6y = e^{2t} + e^{3t}$$

General Solution: $y(t) = -\frac{1}{4}e^{2t} + \frac{1}{5}te^{3t} + c_1 e^{-2t} + c_2 e^{3t}$

Mechanical Vibrations

While second-order differential equations with constant coefficients offers a narrow scope of differential equations, studying them is important because they serve as models of many important applications.

We will study motion of a mass on a spring, as its theory is applicable to other real world scenerios.

We will consider a mass hanging vertically from a spring, stretching it downward.

When the mass-spring system is in equilibrium, the mass has two forces acting on it

Weight, w, coming from gravity: $w = mg$

And the opposite force due to the spring, F_s.

By Hooke's Law F_s is proportional to the elongation distance, L, that the spring is stretched

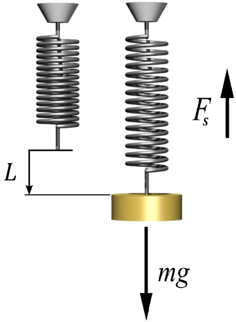
$$F_s = -kL$$

where $k > 0$ is called the *spring constant*.

Since the mass is in equilibrium, we know that the sum of these forces is 0. That is:

$$mg - kL = 0$$

Or, re-writing, $mg = kL$



Images used were adapted from images created by user Svjo on Wikipedia

Mechanical Vibrations

There is not much to study if we leave our mass-spring in equilibrium. Suppose we stretch the mass from equilibrium by length u and set it in motion.

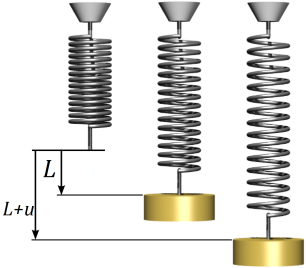
The forces $F_s = -k(L+u)$ and $w = mg$ still act on the mass.

For a mass is in motion, a damping force, F_d, from friction and air resistence will now be acting on the mass, which is proportional to the velocity, u'

$$F_d = -\gamma u'$$

where $\gamma \geq 0$ is called the *damping coefficient*.

Note: If $\gamma = 0$ we call the system *undamped*.



The total forces acting on the mass are:

$$m \cdot u'' = m \cdot a = F = -k(L+u)+mg-\gamma u' = \cancel{-kL}-ku+\cancel{mg}-\gamma u' = -ku-\gamma u'$$

Using that $kL = mg$ we can simplify the right hand.

On the left hand, from Newton's Second Law, we know $F = m \cdot a$

We can put the left hand side in terms of u since acceleration, a, is: $a = u''$

Moving everything to one side, we get the differential equation:

$$mu'' + \gamma u' + ku = 0$$

Images used were adapted from images created by user Svjo on Wikipedia

Mechanical Vibrations - Undamped Case

We studied the motion of a mass on a spring and found that the position of the mass, $u(t)$, could be modeled by the differential equation:

$$mu'' + \gamma u' + ku = 0$$

Where m is the mass, γ is the damping coefficient, and k is the spring constant.
We will look at the undamped case, where $\gamma = 0$
In the undamped case, the differential equations reduces to:

$$mu'' + ku = 0$$

Let's solve this diff. eq. to understand the mass's motion in this case.
We start by solving the characteristic equation:

$$mr^2 + k = 0$$

We'll solve for r by subtracting k and dividing by m to get: $r^2 = -\frac{k}{m}$
Since $k, m > 0$, taking the square root gives the complex solutions:

$$r = \pm\sqrt{-\frac{k}{m}} = \pm i\sqrt{\frac{k}{m}}$$

Letting $\omega_o = \sqrt{\frac{k}{m}}$, we have the $r = \pm i\omega_o$ and get the solutions:

$$u(t) = A\cos(\omega_o t) + B\sin(\omega_o t)$$

For constants A, B that can be determined by the starting position, $u(0)$, and velocity, $u'(0)$.

Mechanical Vibrations - Undamped Case

In the undamped case, the position of a mass on a spring is modeled by:

$$mu'' + ku = 0$$

The solutions, in terms of $\omega_o = \sqrt{\frac{k}{m}}$ and constants A, B are:

$$u(t) = A\cos(\omega_o t) + B\sin(\omega_o t)$$

Since $\cos(\omega_o t)$ and $\sin(\omega_o t)$ are both periodic functions with a period of $\frac{2\pi}{\omega_o}$, it follows that $u(t)$ is a periodic function with **period** $\frac{2\pi}{\omega_o}$, where ω_o is called the **natural frequency**.

It can be useful to write this solution as a single periodic function.
We can do this by making the substitution in our constants:

$$A = R\cos(\delta) \qquad R^2 = R^2\cos^2(\delta) + R^2\sin^2(\delta) = A^2 + B^2$$

$$B = R\sin(\delta) \qquad \tan(\delta) = \frac{R\sin(\delta)}{R\cos(\delta)} = \frac{B}{A}$$

Doing this allows us to rewrite our solutions as:

$$u(t) = R\cos(\delta)\cos(\omega_o t) + R\sin(\delta)\sin(\omega_o t)$$

Factoring out R and using the difference of angles formula:

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

with $\alpha = \omega_o t$ and $\beta = \delta$, we get:

$$u(t) = R\cos(\omega_o t - \delta)$$

In this form, we can still see the period, $\frac{2\pi}{\omega_o}$

We can, also, see the **amplitude** of the mass's position, R.

Mechanical Vibrations - Undamped Example

Example: Consider a mass weighing 16 lb that elongates a spring by 2 feet. We stretch it an addition 1 foot and set it in motion with an initial velocity of $2ft/sec$, causing the mass to oscillate up and down without damping. Let $u(t)$ be the position in feet of the mass t seconds after it is released.
Set up the IVP modeling the position of the mass and solve it to find $u(t)$.
Then find the amplitude of $u(t)$
This undamped case can be modeled by the differential equation:

$$mu'' + ku = 0$$

To find the mass m, we will use that the weight $16 = m \cdot g$ where $g = 32ft/sec^2$, thus $m = \frac{16}{32} = \frac{1}{2}$
The elongation length $L = 2$ with the equation $w = kL$ gives: $k = \frac{16}{2} = 8$
Putting this together, we have the differential equation:

$$\frac{1}{2}u'' + 8u = 0$$

Which can be simplified, by multiplying by 2, as:

$$u'' + 16u = 0$$

This application, also, includes initial conditions since the mass is extended an additional 1 ft, meaning $u(0) = 1$
Additionally, we have that the initial velocity is: $u'(0) = 2$
Together, this gives us the initial value problem:

$$u'' + 16u = 0 \qquad \text{with } u(0) = 1 \text{ and } u'(0) = 2$$

Mechanical Vibrations - Undamped Example

Example: Find the Solution of the IVP:

$$u'' + 16u = 0 \qquad\qquad \text{with } u(0) = 1 \text{ and } u'(0) = 2$$

To start, we find the gen. sol., by solving the char. eq.: $r^2 + 16 = 0$
Subtracting 16 and taking the square roots gives the complex roots: $r = \pm 4i$
Using the root $r = 4i$, we have that the General Solution is:

$$u(t) = A\cos(4t) + B\sin(4t)$$

Now that we have the Gen. Sol., we need to impose the Initial Conditions.
$1 = u(0) = A\cos(4 \cdot 0) + B\sin(4 \cdot 0) = A$
To use $u'(0) = 2$, we need to compute the derivative:

$$u' = -4A\sin(4t) + 4B\cos(4t)$$

$2 = u'(0) = -4A\sin(4 \cdot 0) + 4B\cos(4 \cdot 0) = 4B$
Using that $A = 1$ and $B = \frac{1}{2}$, the position of the mass is given by:

$$u = \cos(4t) + \frac{1}{2}\sin(4t)$$

Using $R = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$ and $\delta = \arctan\left(\frac{1}{2}\right)$, we get:

$$u(t) = \frac{\sqrt{5}}{2}\cos\left(4t - \arctan\left(\frac{1}{2}\right)\right)$$

In this form, we can see that the amplitude is $R = \frac{\sqrt{5}}{2}$

Mechanical Vibrations - Example

Example: Consider a mass weighing 64 lb that elongates a spring by 4 feet. We stretch it an addition 15 inches and let it go, causing the position of the mass to oscillate up and down with a damping coefficient of $\gamma = 8 lb \cdot sec/ft$. Let $u(t)$ be the position in feet of the mass t seconds after it is released. Set up the IVP modeling the position of the mass and solve it to find $u(t)$.

The differential equation modeling u is given by:

$$mu'' + \gamma u' + ku = 0$$

We are given $\gamma = 8$.

To find the mass m, we will use that the weight $64 = m \cdot g$ where $g = 32 ft/sec^2$, thus $m = \frac{64}{32} = 2$

The elongation length $L = 4$ with the equation $w = kL$ gives: $k = \frac{64}{4} = 16$

Putting this together, we have the differential equation:

$$2u'' + 8u' + 16u = 0$$

This application, also, includes initial conditions since the mass is extended an additional 15 inches $= 1.25$ ft, meaning $u(0) = 1.25$

Since the mass is released, we have that the initial velocity is: $u'(0) = 0$

Together, this gives us the initial value problem:

$$2u'' + 8u' + 16u = 0 \qquad \text{with } u(0) = 1.25 \text{ and } u'(0) = 0$$

Mechanical Vibrations - Example

Example: Find the Solution of the IVP:

$$2u'' + 8u' + 16u = 0 \quad \text{with } u(0) = 1.25 \text{ and } u'(0) = 0$$

To start, we find the gen. sol., by solving the char. eq.: $2r^2 + 8r + 16 = 0$
This can't be factored, so we can use the quadratic formula to find the roots:

$$r_{1,2} = \frac{-8 \pm \sqrt{8^2 - 4 \cdot 2 \cdot 16}}{2 \cdot 2} = \frac{-8 \pm \sqrt{-64}}{4} = \frac{-8 \pm 8i}{4} = -2 \pm 2i$$

Using the root $r = -2 + 2i$, we have that the General Solution is:

$$u = c_1 e^{-2t} \cdot cos(2t) + c_2 e^{-2t} \cdot sin(2t)$$

Now that we have the Gen. Sol., we need to impose the Initial Conditions.

$$1.25 = u(0) = c_1 \underbrace{e^{-2\cdot 0}}_{=1} \underbrace{cos(2 \cdot 0)}_{=1} + c_2 \underbrace{e^{-2\cdot 0}}_{=1} \underbrace{sin(2 \cdot 0)}_{=0} = c_1 \qquad \Leftrightarrow c_1 = 1.25$$

To use $u'(0) = 0$, we need to compute the derivative using the product rule:

$$u' = -2c_1 e^{-2t} \cos(2t) - 2c_1 e^{-2t} \sin(2t) - 2c_2 e^{-2t} \sin(2t) + 2c_2 e^{-2t} \cos(2t)$$

Evaluating $u'(0)$, we get: $0 = u'(0) = -2c_1 + 2c_2 = -2.5 + 2c_2 \Leftrightarrow c_2 = 1.25$
Conclusion: the position of the mass is given by:

$$u = 1.25 e^{-2t} \cdot cos(2t) + 1.25 e^{-2t} \cdot sin(2t)$$

Mechanical Vibrations - Example

We can analyze our solution further:

$$u = 1.25e^{-2t} \cdot cos(2t) + 1.25e^{-2t} \cdot sin(2t)$$

Notice that we can factor out e^{-2t} to get:

$$u = e^{-2t} \cdot (1.25\cos(2t) + 1.25\sin(2t))$$

And rewrite the right hand factor in the solution using:

$R = \sqrt{1.25^2 + 1.25^2} = \sqrt{\frac{25}{8}}$ $\qquad$ $\delta = \arctan\left(\frac{1.25}{1.25}\right) = \arctan(1) = \frac{\pi}{4}$

This gives the solution in the form:

$$u = e^{-2t} \cdot \sqrt{\frac{25}{8}} \cos\left(2t - \frac{\pi}{4}\right)$$

In this form, we can see the solution is nearly periodic, but with a decreasing amplitude due to the factor e^{-2t}

That is, the mass will oscillate periodically with a decreasing elongation from its equilibrium position. This decreased elongation is due to the damping in the system.

Mechanical Vibrations - Damping Analysis

Now that we have done an example modeling a spring-mass system, let's look at the differential equation with parameters $m > 0$, $\gamma \geq 0$, $k > 0$:

$$mu'' + \gamma u' + ku = 0$$

We can solve this by looking at the characteristic equation: $mr^2 + \gamma r + k = 0$

We can find the roots using the quadratic formula: $r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$

We saw, when studying second-order homog. diff. eq, that the solutions will depend on whether we have 2 real roots, a repeated root, or complex roots.

Whether the discriminant $\gamma^2 - 4mk$ is positive, zero, or negative will determine which type of solution occurs.

Case 1 $\gamma^2 - 4mk > 0$: The general solution is: $u = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Since $m, k > 0$ we have that $\gamma^2 - 4mk < \gamma^2$ and thus $\sqrt{\gamma^2 - 4mk} < \gamma$

From here, we can conclude that $-\gamma \pm \sqrt{\gamma^2 - 4mk} < 0$ and thus both $r_{1,2} < 0$.

Notice here that, since $r_{1,2} < 0$, the solution $u = c_1 e^{r_1 t} + c_2 e^{r_2 t} \to 0$ as $t \to \infty$

Since this solution does not lead to the mass oscillating up and down, we call this system *overdamped*.

Mechanical Vibrations - Damping Analysis

The roots of the characteristic equation are given by: $r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$

Case 2 $\gamma^2 - 4mk < 0$: The general solution is:

$u = c_1 e^{\frac{-\gamma}{2m}t} \sin(\mu t) + c_2 e^{\frac{-\gamma}{2m}t} \cos(\mu t)$

where $\mu = \frac{\sqrt{4mk - \gamma^2}}{2m}$ is the imaginary part of the roots.

If $\gamma \gneq 0$ we have that $\frac{-\gamma}{2m} < 0$

From here, we can conclude that $e^{\frac{-\gamma}{2m}t} \to 0$ as $t \to \infty$

And thus, the solution $u = c_1 e^{\frac{-\gamma}{2m}t} \sin(\mu t) + c_2 e^{\frac{-\gamma}{2m}t} \cos(\mu t) \to 0$ as $t \to \infty$

Sine and Cosine cause this solution to oscillate up and down with smaller and smaller amplitude, as we would expect from a mass on a spring.

We call this system *underdamped*.

Note: If $\gamma = 0$ there is no damping, and $e^{\frac{-\gamma}{2m}t} = 1$, and the mass will oscillate forever.

Case 3 $\gamma^2 - 4mk = 0$: The general solution is: $u = c_1 e^{\frac{-\gamma}{2m}t} + c_2 t e^{\frac{-\gamma}{2m}t}$

As in the overdamped case, both terms will tend towards 0 as $t \to \infty$

We call this system *critically damped*.

Notice that in all cases where there is damping, i.e. $\gamma > 0$, the solution to the homogeneous diff. eq. tends towards its rest position $u(t) = 0$ as $t \to \infty$.

Forced Mechanical Vibrations

So far, we have studied the case of a mass-spring system w/ no outside forces acting on it

However, there could be an outside force, given by the function $g(t)$, acting on the mass.

In this case, the differential equation modeling the position of the mass, $u(t)$, is given by

$$mu'' + \gamma u' + ku = g(t)$$

Notice here that this system is modelled by a nonhomogeneous diff. eq.

In this case, the solutions will be given by:

$$u(t) = c_1 u_1(t) + c_2 u_2(t) + u_p(t)$$

where $c_1 u_1(t) + c_2 u_2(t)$ is the general solution of the associated homogeneous differential equation we found and $u_p(t)$ is a particular solution to the nonhomog. diff. eq.

We found, in our analysis of the damped mass-spring system with no outside forcing, that $c_1 u_1(t) + c_2 u_2(t) \to 0$ as $t \to \infty$

Considering that here to analyze the long-term behavior of the forced mass-spring system:

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} \overset{\to 0}{c_1 u_1(t) + c_2 u_2(t)} + u_p(t) = \lim_{t\to\infty} u_p(t)$$

Since $u(t) \to u_p(t)$ as $t \to \infty$ we call $u_p(t)$ a *steady-state solution*.

And $c_1 u_1(t) + c_2 u_2(t)$ is called the *transient solution*.

Forced Mechanical Vibrations - Resonance

Consider the class of undamped oscillators with a periodic forcing function:

$$mu'' + ku = F_o \cos(\omega t) \text{ with } F_o = F(0). \text{ the initial force}$$

where ω is the frequency of the forcing function.

To find solutions to this nonhomogeneous diff. eq., we first look at the associated homogenous diff. eq.:

$$mu'' + ku = 0$$

We found the solutions, in terms of $\omega_o = \sqrt{\frac{k}{m}}$ and constants c_1, c_2 are:

$$u(t) = c_1 \cos(\omega_o t) + c_2 \sin(\omega_o t)$$

Given the form of the nonhomogeneous function, $g(t) = F_o \cos(\omega t)$, we know that particular solution will have the form:

$$u_p(t) = A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$$

Where we are defining $R = \sqrt{A^2 + B^2}$ and $\tan(\delta) = \frac{B}{A}$

Note: This is not the particular solution if $\omega = \omega_o$, that is the frequency of the forcing function is the same as the natural frequency of the mass-spring system.

In the case where $\omega = \omega_o$, the particular solution has the form:

$$u_p(t) = t\left(A\cos(\omega t) + B\sin(\omega t)\right) = Rt\cos(\omega t - \delta)$$

In this case, as t increases, so does the amplitude, Rt, of the particular solution; and thus, all solutions to the differential equation.

This phenomenon that happens when the forcing frequency is equal to the natural frequency, $\omega = \omega_o$, is called **resonance** and is important to consider.

Introduction to Laplace Transformations

Thus far, most of our theory and techniques of solving differential equations has relied on the functions involved being continuous.
However, for some applications we may have discontinuous functions involved.
So, we will need different techniques for solving such differential equations.
The method we will learn is the method of Laplace Transformations.
Integrals have a smoothing effect on functions. For example, when we differentiate the continuous function $y = |x|$ we get a discontinuous function:

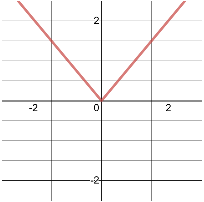


Figure: $y = |x|$

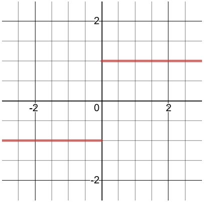


Figure: $y' = \frac{d}{dx}(|x|)$

In reverse, integrating y' would smooth out that discontinuity.
Note: While the motivation for Laplace Transforms is to be able to solve differential equations involving discontinuous functions, it can be used to solve many differential equations.

Introduction to Laplace Transformations

We begin by defining the Laplace transform for a function $f(t)$

We write the Laplace Transform as $F(s) = \mathcal{L}\{f(t)\}$, and it is defined as:

$$\mathcal{L}\{f(t)\} := \int_0^\infty e^{-st} \cdot f(t)dt$$

Since this is an improper integral, $F(s)$ is only defined at values of s so that the integral converges.

Theorem: Suppose that f is a piecewise continuous function and there exist constants K, a, and M such that $f(t)$ is bound by:

$$|f(t)| \leq Ke^{at} \text{ for } t \geq M$$

then $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$

Proof: We must show the integral defining $F(s)$ converges for $s > a$:

$$\int_0^\infty e^{-st} \cdot f(t)dt = \underbrace{\int_0^M e^{-st} f(t)dt}_{finite} + \underbrace{\int_M^\infty e^{-st} f(t)dt}_{\leq \int_M^\infty e^{-st} Ke^{at} dt}$$

For $s > a$, we have $(a - s) < 0$ and thus

$$\int_M^\infty e^{-st} Ke^{at} dt = K \int_M^\infty e^{(a-s)t} dt$$

converges. So, we can conclude that $F(s)$ exists for $s > a$.

Computation of Laplace Transformations of 1

We will compute the Laplace transform of $f(t) = 1$

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty e^{-st} \cdot 1 dt \\
&= \int_0^\infty e^{(-s)t} dt \\
&= \lim_{b\to\infty} \frac{1}{(-s)} e^{(-s)\cdot t}\Big|_0^b \\
&= \lim_{b\to\infty} \underbrace{\cancel{\frac{1}{(-s)} e^{(-s)\cdot b}}}_{\text{if } -s<0} \overset{\nearrow 0}{} - \frac{1}{(-s)} \underbrace{e^{(-s)\cdot 0}}_{1} = \frac{1}{s}
\end{aligned}$$

So, we can conclude that: $\mathcal{L}\{1\} = \dfrac{1}{s}$ for $s > 0$

Computation of Laplace Transformations of e^{at}

We will compute the Laplace transform of $f(t) = e^{at}$

$$\mathcal{L}\left\{e^{at}\right\} = \int_0^{\infty} e^{-st} \cdot e^{at} dt$$

$$= \int_0^{\infty} e^{(a-s)t} dt$$

$$= \lim_{b\to\infty} \frac{1}{(a-s)} e^{(a-s)\cdot t}\Big|_0^b$$

$$= \lim_{b\to\infty} \underbrace{\cancel{\frac{1}{(a-s)} e^{(a-s)\cdot b}}}_{\text{if } a-s<0} \xrightarrow{} 0 - \frac{1}{(a-s)} \underbrace{e^{(a-s)\cdot 0}}_{1} = \frac{1}{s-a}$$

So, we can conclude that: $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$ for $s > a$

Computation of Laplace Transformations of t

We will compute the Laplace transform of $f(t) = t$

$$\mathcal{L}\left\{t\right\} = \int_0^\infty e^{-st} \cdot t\,dt = \int_0^\infty t \cdot e^{-st} dt$$

We can compute the antiderivative using integration by parts to find:

$$\int t \cdot e^{-st} dt = \frac{-1}{s} \cdot t \cdot e^{-s \cdot t} - \frac{1}{s^2} e^{-s \cdot t}$$

Evaluating this at the bounds to find the Laplace Transform yields:

$$\begin{aligned}
\mathcal{L}\left\{t\right\} &= \lim_{b\to\infty} \frac{-1}{s} \cdot t \cdot e^{-s \cdot t} - \frac{1}{s^2} e^{-s \cdot t} \Big|_0^b \\
&= \lim_{b\to\infty} \underbrace{\cancelto{0}{\frac{-1}{s} \cdot b \cdot e^{-s \cdot b} - \frac{1}{s^2} e^{-s \cdot b}}}_{\text{if } -s<0} - \left(\cancelto{0}{-\frac{1}{s} \cdot 0 \cdot e^{-s \cdot 0}} - \frac{1}{s^2} \underbrace{e^{-s \cdot 0}}_{1} \right) \\
&= \frac{1}{s^2}
\end{aligned}$$

So, we can conclude that: $\mathcal{L}\left\{t\right\} = \dfrac{1}{s^2}$ for $s > 0$

Computation of Laplace Transformations of $sin\,(\alpha t)$

We will compute the Laplace transform of $f(t) = sin\,(\alpha t)$

$$\mathcal{L}\left\{sin\,(\alpha t)\right\} = F(s) = \int_0^\infty e^{-st} \cdot sin\,(\alpha t) dt$$

Integrating by parts, using $u = e^{-st}$ and $dv = sin\,(\alpha t)dt$, we get:
$du = -s \cdot e^{-st}dt$ and $v = -\frac{1}{\alpha}cos(\alpha t)$ which gives that our improper integral is:

$$F(s) = \lim_{b\to\infty}\left(-\frac{e^{-st}cos(\alpha t)}{\alpha}\Bigg|_0^b - \frac{s}{\alpha}\int_0^b e^{-st}cos(\alpha t)dt\right)$$

$$F(s) = \lim_{b\to\infty}\left(\cancelto{0}{-\frac{\cancelto{0}{e^{-sb}}cos(\alpha b)}{\alpha}} + \frac{\overbrace{e^{-s\cdot 0}}^{=1}\overbrace{cos(\alpha\cdot 0)}^{=1}}{\alpha} - \frac{s}{\alpha}\int_0^b e^{-st}cos(\alpha t)dt\right)$$

$$= \frac{1}{\alpha} - \frac{s}{\alpha}\int_0^\infty e^{-st}cos(\alpha t)dt = \frac{1}{\alpha} - \frac{s^2}{\alpha^2}\underbrace{\int_0^\infty e^{-st}sin(\alpha t)dt}_{=F(s)}$$

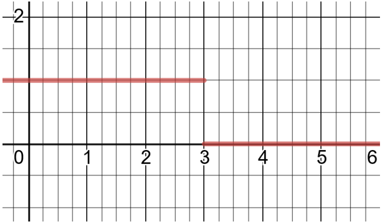
Solving for $F(s)$ in the equation: $F(s) = \frac{1}{\alpha} - \frac{s^2}{\alpha^2} \cdot F(s)$

We can conclude that: $\mathcal{L}\left\{sin\,(\alpha t)\right\} = F(s) = \dfrac{\alpha}{s^2+\alpha^2}$ for $s > 0$

Laplace Transformations of a discontinuous function

We will compute the Laplace transform of the discontinuous function:

$$f(t) = \begin{cases} 1 & t \leq 3 \\ 0 & t > 3 \end{cases}$$



0

$$\mathcal{L}\left\{f(t)\right\} = \int_0^{\infty} e^{-st} \cdot f(t) dt = \int_0^{3} e^{-st} \cdot \underbrace{f(t)}_{=1} dt + \int_3^{\infty} e^{-st} \cdot \underbrace{f(t)}_{=0} dt$$

$$= \int_0^{3} e^{-st} dt = \frac{1}{-s} e^{-s \cdot t}\Big|_0^3 = \frac{1}{-s} e^{-s \cdot 3} - \frac{1}{-s} \underbrace{e^{-s \cdot 0}}_{=1}$$

So, we can conclude that: $\mathcal{L}\left\{f(t)\right\} = \dfrac{1}{s} - \dfrac{1}{s} e^{-3s}$

Linearity of Laplace Transformations

In working with Laplace Transforms, and differential equations in general, we often have more than one term involved.

So, we want to be able to compute the Laplace Transform of linear combinations of functions.

i.e. for functions, $f_1(t)$ and $f_2(t)$ with constants c_1 and c_2, we want to compute:

$$\mathcal{L}\left\{c_1 \cdot f_1(t) + c_2 \cdot f_2(t)\right\} = \int_0^\infty e^{-st} \cdot (c_1 \cdot f_1(t) + c_2 \cdot f_2(t))\, dt$$

Due to the linearity of integrals, we can rewrite this as:

$$\begin{aligned} &= c_1 \cdot \int_0^\infty e^{-st} \cdot f_1(t) dt + c_2 \cdot \int_0^\infty e^{-st} \cdot f_2(t) dt \\ &= c_1 \cdot \mathcal{L}\left\{f_1(t)\right\} + c_2 \cdot \mathcal{L}\left\{f_2(t)\right\} \end{aligned}$$

as long as both improper integrals converge.

Both integrals converge for values of s where $\mathcal{L}\left\{f_1(t)\right\}$ and $\mathcal{L}\left\{f_2(t)\right\}$ exist.

In other words, if $\mathcal{L}\left\{f_1(t)\right\}$ exists for $s > s_1$ and $\mathcal{L}\left\{f_2(t)\right\}$ exists for $s > s_2$ then both improper integrals converge for $s > max(s_1, s_2)$ and thus:

$$\mathcal{L}\left\{c_1 \cdot f_1(t) + c_2 \cdot f_2(t)\right\} = c_1 \cdot \mathcal{L}\left\{f_1(t)\right\} + c_2 \cdot \mathcal{L}\left\{f_2(t)\right\} \quad \text{for } s > max(s_1, s_2).$$

Linearity of Laplace Transformations

Example: Compute the Laplace Transform:

$$\mathcal{L}\left\{-2 \cdot t + 3 \cdot e^{2t}\right\}$$

Using the Linearity of Laplace Transforms we can rewrite this Laplace Transform as:

$$\mathcal{L}\left\{-2 \cdot t + 3 \cdot e^{2t}\right\} = -2 \cdot \mathcal{L}\left\{t\right\} + 3 \cdot \mathcal{L}\left\{e^{2t}\right\}$$

We previously computed $\mathcal{L}\left\{t\right\} = \frac{1}{s^2}$ and computed $\mathcal{L}\left\{e^{2t}\right\} = \frac{1}{s-2}$.

Using these, we get:

$$= -2 \cdot \frac{1}{s^2} + 3 \cdot \frac{1}{s-2}$$

Note: since $\mathcal{L}\left\{t\right\}$ is defined for $s > 0$ and $\mathcal{L}\left\{e^{2t}\right\}$ is defined for $s > 2$, we know that $\mathcal{L}\left\{-2 \cdot t + 3 \cdot e^{2t}\right\}$ is defined for $s > 2$

Differential Equations with Laplace Transformations

Now that we've defined the Laplace Transform and computed it for some functions, we will look at how to solve differential equations using them.
The defining trait of a differential equation is that it involves a derivative.
So, we need to understand the Laplace Transform of a derivative.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = \int_0^\infty e^{-st} \cdot \frac{dy}{dt}\, dt$$

Computing this using the integration by parts with $u = e^{-st}$ and $dv = \frac{dy}{dt}dt$
We have $du = -se^{-st}dt$ and using the FTC, $v = y$
Now we can rewrite our Laplace Transform as:

$$\begin{aligned}
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= \lim_{b\to\infty} e^{-st}y\Big|_{t=0}^{b} - (-s)\int_0^b e^{-st}y\,dt \\
&= \lim_{b\to\infty} \underbrace{\cancelto{0}{e^{-s\cdot b}}y(b)}_{\to 0} - \underbrace{\cancelto{1}{e^{-s\cdot 0}}y(0)}_{=y(0)} + s\underbrace{\int_0^b e^{-st}y\,dt}_{=\mathcal{L}\{y\}} \\
&= s\cdot\mathcal{L}\{y\} - y(0)
\end{aligned}$$

Thus $\mathcal{L}\left\{\dfrac{dy}{dt}\right\} = s\cdot\mathcal{L}\{y\} - y(0)$ for values of s so that $\mathcal{L}\{y\}$ exists.

Differential Equations with Laplace Transformations

Example: Find the solution of the Initial Value Problem:

$$y' = 3y \qquad\qquad \text{with } y(0) = 4$$

Note: This DiffEq is separable and linear, so we can solve it with older methods

We may even recognize the solution of such an equation to be: $y(t) = y_o \cdot e^{3t}$

And since $y_o = y(0) = 4$, we know that the solution must be $y(t) = 4 \cdot e^{3t}$

Let's revisit this after we compute the solution using Laplace Transforms.

Since both sides of this equation are equal, their Laplace Transforms are equal:

$$s \cdot \mathcal{L}\{y\} - \underbrace{y(0)}_{=4} = \mathcal{L}\{y'\} = \mathcal{L}\{3y\} = 3\mathcal{L}\{y\}$$

So, we're left with the equation: $s \cdot \mathcal{L}\{y\} - 4 = 3 \cdot \mathcal{L}\{y\}$

Adding 4 and subtracting $3 \cdot \mathcal{L}\{y\}$ from both sides yields: $(s-3)\,\mathcal{L}\{y\} = 4$

And thus we have $\mathcal{L}\{y\} = \dfrac{4}{s-3} = 4 \cdot \dfrac{1}{s-3} = 4 \cdot \mathcal{L}\left\{e^{3t}\right\} = \mathcal{L}\left\{4e^{3t}\right\}$

We now know $\mathcal{L}\{y\}$, but we want to know y itself.

We need to be able to work backwards to find y from $\mathcal{L}\{y\}$

To do this here, we need to make the observation that: $\mathcal{L}\left\{e^{3t}\right\} = \frac{1}{s-3}$

This leaves us with: $\mathcal{L}\{y\} = \mathcal{L}\left\{4e^{3t}\right\}$, which lines up the sol'n: $y = 4 \cdot e^{3t}$

Differential Equations with Laplace Transformations

We saw that we can use Laplace Transforms to solve an Initial Value Problem.
Let's take a look at this process with a generic differential equation:

$$\frac{dy}{dt} = f(t, y)$$
$$\Downarrow$$
$$s \cdot \mathcal{L}\{y\} - y(0) = \mathcal{L}\left\{\frac{dy}{dt}\right\} = \mathcal{L}\{f(t, y)\}$$

We start by taking the Laplace Transform of both sides

We computed that the Laplace Transform of $\mathcal{L}\left\{\frac{dy}{dt}\right\} = s \cdot \mathcal{L}\{y\} - y(0)$.

The right hand side will depend on the function(s) involved, but will, also, end up written in terms of $\mathcal{L}\{y\}$.
Thus, we will have an algebraic equation for $\mathcal{L}\{y\}$
Note that an *algebraic* equation for $\mathcal{L}\{y\}$ is much easier to solve for $\mathcal{L}\{y\}$ than the *differential* equation we started with that we needed to solve for y.
Solving this algebraic equation for $\mathcal{L}\{y\}$, we will be left with:

$$\mathcal{L}\{y\} = F(s)$$

Notice that this does not give us $y(t)$, the solution to the Diff. Eq., directly!
We need to be able to find $y(t)$ **by knowing its Laplace Transform,** $\mathcal{L}\{y\}$
This process of finding $y(t)$ from $\mathcal{L}\{y\}$ is called the *Inverse Laplace Transform*
Taking the inverse is, typically, the most difficult step in this process.
Note: This inverse is well-defined, meaning that if $\mathcal{L}\{y_1\} = \mathcal{L}\{y_2\}$ then $y_1 = y_2$

Diff Eq w/ order 2+ using Laplace Transforms

We saw earlier that the Laplace Transform of y' is:

$$\underbrace{\mathcal{L}\{y'\}} = \underbrace{s \cdot \mathcal{L}\{y\} - y(0)}$$

What about higher order differential equations?

We will use what we know about $\mathcal{L}\{y'\}$ to find $\mathcal{L}\{y''\}$

Since $y'' = (y')'$ we can compute it's Laplace Transform as:

$$\mathcal{L}\{y''\} = \mathcal{L}\{(y')'\} = s \cdot \underbrace{\mathcal{L}\{y'\}} - y'(0) = s \cdot \left(\underbrace{s \cdot \mathcal{L}\{y\} - y(0)}\right) - y'(0)$$

So, we can conclude that: $\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - s \cdot y(0) - y'(0)$

Note: We can repeat this process for higher level derivatives as well.

In General:

$$\mathcal{L}\left\{y^{(n)}\right\} = s^n\mathcal{L}\{y\} - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - sy^{(n-2)}(0) - y^{(n-1)}(0)$$

Differential Equations with Laplace Transformations

We saw earlier that the Laplace Transform of $y(t) = sin\left(\alpha t\right)$ is:

$$\mathcal{L}\left\{sin\left(\alpha t\right)\right\} = \frac{\alpha}{s^2 + \alpha^2}$$

We can find the Laplace Transform of $cos\left(\alpha t\right)$ in a similar fashion (recall that it required integration by parts twice).

Instead, we will use what we know about $\mathcal{L}\left\{sin\left(\alpha t\right)\right\}$ to find $\mathcal{L}\left\{cos\left(\alpha t\right)\right\}$ in a clever way.

We will use that $y(t) = sin\left(\alpha t\right)$ satisfies the differential equation:

$$y' = \alpha cos\left(\alpha t\right) \text{ with } y(0) = sin(\alpha \cdot 0) = 0$$

Taking the Laplace Transform of both sides yields:

$$s \cdot \tfrac{\alpha}{s^2+\alpha^2} = s \cdot \mathcal{L}\left\{y\right\} - \underbrace{y(0)}_{=0} = \mathcal{L}\left\{y'\right\} = \mathcal{L}\left\{\alpha cos\left(\alpha t\right)\right\} = \alpha \cdot \mathcal{L}\left\{cos\left(\alpha t\right)\right\}$$

This leaves us with the equation:

$$\tfrac{s \cdot \alpha}{s^2+\alpha^2} = \alpha \cdot \mathcal{L}\left\{cos\left(\alpha t\right)\right\}$$

We can solve this algebraic equation for $\mathcal{L}\left\{cos\left(\alpha t\right)\right\}$ by dividing by α.

So, we can conclude that: $\mathcal{L}\left\{cos\left(\alpha t\right)\right\} = \dfrac{s}{s^2 + \alpha^2}$

2nd Order Diff Eq with Laplace Ex 1

Example: Solve the initial value problem:

$$y'' + y' - 12y = 0 \quad \text{with } y(0) = 2 \text{ and } y'(0) = 1$$

Solution: **Taking the Laplace Transform of both sides**, we get:

$$\underbrace{\mathcal{L}\{y''\}}_{s^2\cdot\mathcal{L}\{y\}-s\cdot 2-1} + \underbrace{\mathcal{L}\{y'\}}_{s\cdot\mathcal{L}\{y\}-2} - 12\mathcal{L}\{y\} = \mathcal{L}\{y'' + y' - 12y\} = \mathcal{L}\{0\} = 0$$

Cleaning this equation up, we have:

$$\underbrace{s^2 \cdot \mathcal{L}\{y\} - s \cdot 2 - 1} + \underbrace{s \cdot \mathcal{L}\{y\} - 2} - 12\mathcal{L}\{y\} = 0$$

To solve for $\mathcal{L}\{y\}$, we collect all terms involving $\mathcal{L}\{y\}$ on one side to get:

$$\left(s^2 + s - 12\right) \cdot \mathcal{L}\{y\} = 2s + 3 \Rightarrow \mathcal{L}\{y\} = \frac{2s+3}{s^2+s-12} = \frac{2s+3}{(s-3)(s+4)}$$

We need to find the Laplace Inverse, $y = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\}$, for the above $\mathcal{L}\{y\}$

In our 1st-order example we used that $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

To use this inverse here, we need to factor $s^2 + s - 12 = (s-3)(s+4)$

Then use partial fractions to split $\mathcal{L}\{y\} = \dfrac{2s+3}{(s-3)(s+4)} = \dfrac{9/7}{s-3} + \dfrac{5/7}{s+4}$

Writing it this way, we have: $y = \mathcal{L}^{-1}\left\{\dfrac{9/7}{s-3} + \dfrac{5/7}{s+4}\right\} = \dfrac{9}{7}e^{3t} + \dfrac{5}{7}e^{-4t}$

2nd Order Diff Eq with Laplace Ex 2

Example: Solve the initial value problem:

$$y'' + y = e^{2t} \qquad \text{with } y(0) = 0 \text{ and } y'(0) = 0$$

Solution: **Taking the Laplace Transform of both sides**, we get:

$$s^2\mathcal{L}\{y\} + \mathcal{L}\{y\} = \underbrace{\mathcal{L}\{y''\}}_{s^2\cdot\mathcal{L}\{y\}-s\cdot 0-0} + \mathcal{L}\{y\} = \mathcal{L}\{y'' + y\} = \mathcal{L}\left\{e^{2t}\right\} = \frac{1}{s-2}$$

This leaves us with the equation: $\left(s^2+1\right)\mathcal{L}\{y\} = s^2\mathcal{L}\{y\} + \mathcal{L}\{y\} = \frac{1}{s-2}$

To solve for $\mathcal{L}\{y\}$, we factor out $\mathcal{L}\{y\}$ on the left and divide by $\left(s^2+1\right)$:

$$\mathcal{L}\{y\} = \frac{1}{(s^2+1)\cdot(s-2)} = \frac{(-1/5)s-2/5}{s^2+1} + \frac{1/5}{s-2} = \frac{-1}{5}\cdot\frac{s}{s^2+1} - \frac{2}{5}\cdot\frac{1}{s^2+1} + \frac{1}{5}\cdot\frac{1}{s-2}$$

We need to find the Laplace Inverse, $y = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\}$, for the above $\mathcal{L}\{y\}$

In our last example we factored the quadratic and used partial fractions.

Then, for each linear term in the denominator, we used: $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

However, we cannot factor s^2+1. But we can still use partial fractions.

For the term $\frac{1/5}{s-2}$, we can find the inverse Laplace as: $\mathcal{L}^{-1}\left\{\frac{1/5}{s-2}\right\} = \frac{1}{5}e^{2t}$

To find the inverse Laplace for the other term, we need to use:

$\mathcal{L}\{cos(\alpha t)\} = \frac{s}{s^2+\alpha^2}$ and $\mathcal{L}\{sin(\alpha t)\} = \frac{\alpha}{s^2+\alpha^2}$ with $\alpha = 1$

We can conclude that: $y = -\dfrac{1}{5}cos(t) - \dfrac{2}{5}sin(t) + \dfrac{1}{5}e^{2t}$

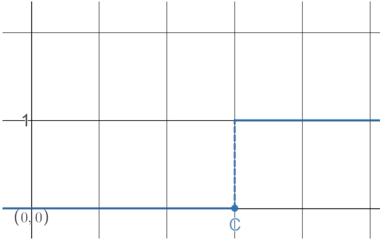
Introduction to Step Functions

We motivated the need for the Method of Laplace Transforms in solving Diff. Eq. by the fact that our older methods failed for discontinuous functions.

Here, we will define a basic discontinuous function: the Step Function

We define the step function, $u_c(t)$ in the following way:

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

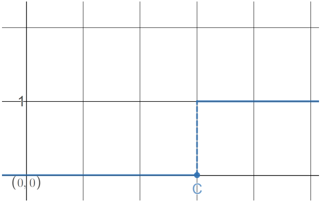


While it is a very basic function, it will be used as a building block of of other discontinuous functions.

Modifications to Step Functions

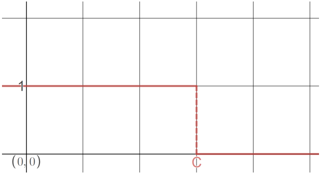
We can make modifications to $u_c(t)$ to build other discontinuous functions

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$



To reverse our step function to "step down", how should we change $u_c(t)$?

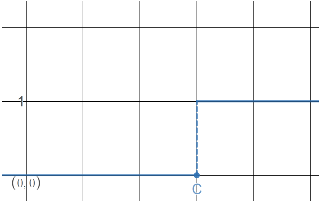
$$1 - u_c(t) = \begin{cases} 1 & t < c \\ 0 & t \geq c \end{cases}$$



Modifications to Step Functions

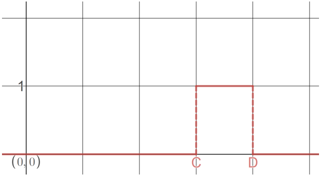
We can make modifications to $u_c(t)$ to build other discontinuous functions

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$



To step up then back down, how should we change $u_c(t)$?

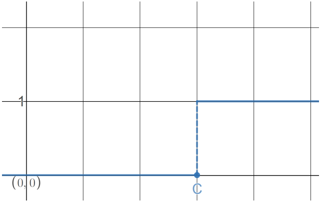
$$u_c(t) - u_d(t) = \begin{cases} 0 & t < c \\ 1 & c < t \leq d \\ 0 & t \geq d \end{cases}$$



Modifications to Step Functions

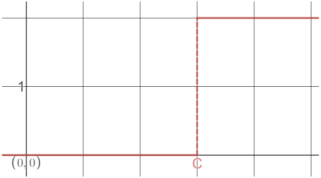
We can make modifications to $u_c(t)$ to build other discontinuous functions

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$



To scale our step, how should we change $u_c(t)$?

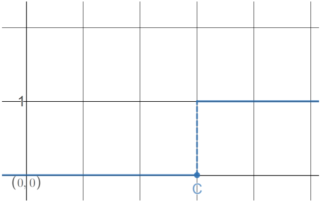
$$2 \cdot u_c(t) = \begin{cases} 0 & t < c \\ 2 & t \geq c \end{cases}$$



Modifications to Step Functions

We can make modifications to $u_c(t)$ to build other discontinuous functions

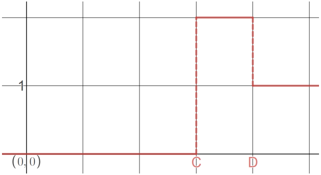
$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$



We can, also, take combinations of these changes, such as having a function that steps up by 2 then back down to 1.

How can we build this from step functions?

$$2 \cdot u_c(t) - u_d(t) = \begin{cases} 0 & t < c \\ 2 & c < t \leq d \\ 1 & t \geq d \end{cases}$$

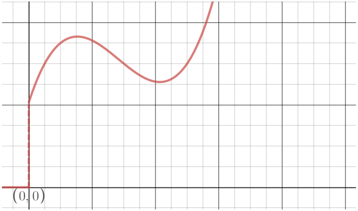


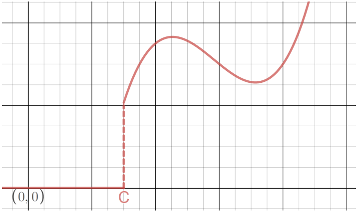
Shifted Functions

Suppose we have a function $f(t)$ modeling a particular process starting at $t = 0$.

Now suppose that we wait to start this process until $t = c$

This will shift the graph of $f(t)$ by c, with the process "off" for $t < c$





We want to alter our function $f(t)$ to get this shifted version.

To do this, we'll combine what we learned about step functions with our understanding of graph shifting from algebra.

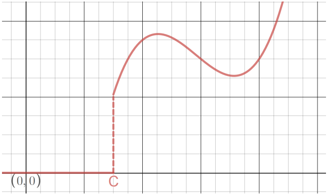
From algebra, $f(t-c)$ is the function $f(t)$ shifted in the t direction by c.

So, this shifted function can be written as:

$$u_c(t) \cdot f(t-c) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c \end{cases}$$

Laplace Transformations of a shifted step function

We will compute the Laplace transform of the shifted function:



$$u_c(t)f(t-c) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c \end{cases}$$

$$\mathcal{L}\left\{u_c(t)f(t-c)\right\} = \int_0^\infty e^{-st}\cdot u_c(t)f(t-c)dt$$

$$= \cancelto{0}{\int_0^c e^{-st}\cdot \underbrace{u_c(t)}_{=0} f(t-c)dt} + \int_c^\infty e^{-st}\cdot \underbrace{u_c(t)}_{=1} f(t-c)dt$$

$$= \int_c^\infty e^{-st}\cdot f(\underbrace{t-c}_{v=t-c})dt = \int_0^\infty \underbrace{e^{-s(v+c)}}_{=e^{-sv}\cdot e^{-sc}}\cdot f(v)dv$$

Substituting, we get $dv = dt$ and the lower bound changes from $t = c$ to $v = 0$
Note that e^{-sc} is constant with respect to v

$$\mathcal{L}\left\{u_c(t)f(t-c)\right\} = e^{-sc}\cdot \underbrace{\int_0^\infty e^{-sv}\cdot f(v)dv} = e^{-sc}\cdot \underbrace{\mathcal{L}\left\{f(t)\right\}}$$

So, we can conclude that: $\mathcal{L}\left\{u_c(t)f(t-c)\right\} = e^{-sc}\cdot \mathcal{L}\left\{f(t)\right\}$
Reinterpreting in terms of the inverse: $\mathcal{L}^{-1}\left\{e^{-sc}\cdot \mathcal{L}\left\{f(t)\right\}\right\} = u_c(t)f(t-c)$

Laplace Transform of a shifted step function - Ex 1

Example: Compute the Laplace Transform of:

$$\mathcal{L}\{u_2(t) \cdot sin(\underbrace{3t - 6}))\} = \mathcal{L}\{u_2(t) \cdot sin(3 \cdot (t-2))\}$$

$$= e^{-2s} \cdot \mathcal{L}\{sin(3t)\}$$

$$= e^{-2s} \cdot \frac{3}{s^2 + 3^2}$$

Solution: We found that the Laplace Transform of a shifted function is:

$\mathcal{L}\{u_c(t)f(\underbrace{t-c})\} = e^{-sc} \cdot \mathcal{L}\{f(t)\}$

We can see, from the step function, that it's shifted by $c = 2$

We need to write the rest of the function in terms of $t - 2$

That is, we need to find a function $f(t)$ so that $f(t-2) = sin(3t-6)$

Writing $3t - 6 = 3 \cdot (t-2)$, we have $sin(3t-6) = sin(3 \cdot (t-2))$

So, using $f(t) = sin(3t)$, our original function, $sin(3t-6) = sin(3 \cdot (t-2))$, is $f(t) = sin(3t)$ shifted by $c = 2$

Inverse Laplace Transform - Ex 1

Example: Compute the Inverse Laplace Transform of:

$$F(s) = \frac{e^{-4s}+1}{s-2}$$

Rewriting this as an Inverse Laplace, we want to compute:

$$\mathcal{L}^{-1}\left\{\frac{e^{-4s}+1}{s-2}\right\} = \mathcal{L}^{-1}\left\{e^{-4s}\cdot\frac{1}{s-2}+\frac{1}{s-2}\right\} = \mathcal{L}^{-1}\left\{e^{-4s}\cdot\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= \underbrace{\mathcal{L}^{-1}\left\{e^{-4s}\cdot\mathcal{L}\left\{e^{2t}\right\}\right\}} + \underbrace{\mathcal{L}^{-1}\left\{\mathcal{L}\left\{e^{2t}\right\}\right\}}$$

$$= \underbrace{u_4(t)\cdot e^{2(t-4)}} + \underbrace{e^{2t}}$$

Noticing that the Laplace function, $F(s)$, includes e^{-4s} should make us aware that we will need a step function with $c = 4$ since:

$$\underbrace{\mathcal{L}^{-1}\left\{e^{-sc}\cdot\mathcal{L}\left\{f(t)\right\}\right\}} = u_c(t)f(t-c)$$

In order to use this, we need e^{-sc} to be a factor (multiplied by some function)
We don't have this, but can get it by splitting the numerator.
Using Linearity, we can split the inverse into its two terms.
In both of these terms, we'll use that $\mathcal{L}\left\{e^{2t}\right\} = \frac{1}{s-2}$

So, we can conclude that: $\mathcal{L}^{-1}\left\{\frac{e^{-4s}+1}{s-2}\right\} = u_4(t)\cdot e^{2(t-4)} + e^{2t}$

Inverse Laplace Transform - Ex 2

Example: Compute the Inverse Laplace Transform of:

$$F(s) = \frac{3e^{-3s}}{s^2+4}$$

Rewriting this as an Inverse Laplace, we want to compute:

$$\mathcal{L}^{-1}\left\{\frac{3e^{-3s}}{s^2+4}\right\} = \underbrace{\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{3}{s^2+4}\right\}} = \underbrace{\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{3}{s^2+2^2}\right\}}$$

$$= u_3(t)\cdot\frac{3}{2}\cdot sin(2(t-3))$$

Noticing that the Laplace function, $F(s)$, includes e^{-3s} should make us aware that we will need a step function with $c = 3$ since:

$$\underbrace{\mathcal{L}^{-1}\left\{e^{-sc}\cdot\mathcal{L}\left\{f(t)\right\}\right\}} = u_c(t)f(t-c)$$

We can factor out e^{-3s} to use this more directly

To compute this Laplace Inverse, we need to find $f(t)$ so that $\mathcal{L}\left\{f(t)\right\} = \frac{3}{s^2+4}$

We can't factor s^2+4, so we use: $\mathcal{L}\left\{sin(\alpha t)\right\} = \frac{\alpha}{s^2+\alpha^2}$ or $\mathcal{L}\left\{cos(\alpha t)\right\} = \frac{s}{s^2+\alpha^2}$

Since the numerator is a constant, we use $\mathcal{L}\left\{sin(\alpha t)\right\} = \frac{\alpha}{s^2+\alpha^2}$ with $\alpha = 2$

We need the numerator to be $\alpha = 2$, so we rewrite $3 = \frac{3}{2}\cdot 2$

Laplace Transform of $e^{ct} \cdot f(t)$

In general, we can't compute the Laplace of a product of functions: $f(t) \cdot g(t)$
We are, however, able to compute the Laplace of a product for certain functions.
Here, we will like at the product where one of the functions is e^{ct}
That is, we will compute the Laplace Transform of:

$$\mathcal{L}\left\{e^{ct} \cdot f(t)\right\} = \int_0^\infty \underbrace{e^{-st} \cdot e^{ct}}_{=e^{-(s-c)t}} \cdot f(t)dt = \int_0^\infty e^{-(s-c)t} \cdot f(t)dt = F(s-c)$$

To do this, we will come back to the definition of the Laplace Transform:

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^\infty e^{-st} \cdot f(t)dt$$

One reason that we can compute the Laplace Transform of product of functions where one is an exponential is that we can combine: $\underbrace{e^{-st} \cdot e^{ct}} = \underbrace{e^{-st+ct}}$

Next, we factor out $-t$ in the exponent, to get: $\underbrace{e^{-st+ct}} = \underbrace{e^{-(s-c)t}}$

Notice that we are *almost* left with the definition of the Laplace Transform.
The difference is that the we have $(s-c)$ rather than s
So, we can conclude that: $\mathcal{L}\left\{e^{ct} \cdot f(t)\right\} = F(s-c)$ where $F(s) = \mathcal{L}\left\{f(t)\right\}$

Laplace Transform of $e^{ct} \cdot f(t)$

Example: Compute the Inverse Laplace Transform of:

$$G(s) = \frac{1}{s^2 + 2s + 2}$$

If we write $s^2 + 2s + 2$ in factored form, we can split $\frac{1}{s^2+2s+2}$

Unfortunately, $s^2 + 2s + 2$ cannot be factored using real numbers.

To proceed, we will need to write $s^2 + 2s + 2$ in vertex form: $(s - h)^2 + k$

Note: The general vertex form is: $a(s - h)^2 + k$ where a is the coefficient of s^2

The process of changing a quadratic in standard form, $s^2 + 2s + 2$, to vertex form is called "completing the square"

Writing $s^2 + 2s + 2$ in vertex form, we get: $s^2 + 2s + 2 = (s + 1)^2 + 1$

Thus: $\frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1} = F(s + 1)$ where $F(s) = \frac{1}{s^2+1} = \mathcal{L}\left\{sin(t)\right\}$

Recall: $\mathcal{L}\left\{e^{ct} \cdot f(t)\right\} = F(s - c)$ where $F(s) = \mathcal{L}\left\{f(t)\right\}$

Using $c = -1$ here, we get: $\frac{1}{s^2+2s+2} = \mathcal{L}\left\{e^{-1t} sin(t)\right\}$

And we can conclude: $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} = e^{-t} sin(t)$

Solving a discontinuous Differential Equation

Example: Solve the Initial Value Problem

$$y'' + 2y' + 2y = u_2(t) \quad \text{with } y(0) = 0 \text{ and } y'(0) = 1$$

Solution: **Taking the Laplace Transform of both sides**, we get:

$$\underbrace{\mathcal{L}\{y''\}}_{s^2\cdot\mathcal{L}\{y\}-s\cdot 0-1} + 2\underbrace{\mathcal{L}\{y'\}}_{s\cdot\mathcal{L}\{y\}-0} + 2\mathcal{L}\{y\} = \mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{u_2(t)\} = \frac{e^{-2s}}{s}$$

Cleaning this equation up, we have:

$$\underbrace{s^2 \cdot \mathcal{L}\{y\} - s \cdot 0 - 1} + 2 \cdot \underbrace{(s \cdot \mathcal{L}\{y\} - 0)} + 2\mathcal{L}\{y\} = \frac{e^{-2s}}{s}$$

To solve for $\mathcal{L}\{y\}$, we collect all terms involving $\mathcal{L}\{y\}$ on one side to get:

$$\left(s^2 + 2s + 2\right) \cdot \mathcal{L}\{y\} = 1 + \frac{e^{-2s}}{s} \Rightarrow \mathcal{L}\{y\} = \frac{1 + \frac{e^{-2s}}{s}}{s^2 + 2s + 2}$$

We need to find the Laplace Inverse, $y = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\}$, for the above $\mathcal{L}\{y\}$

Writing this in two terms allows us to take the Laplace Inverse of each term.

To find the Laplace Inverse of $\frac{\frac{e^{-2s}}{s}}{s^2+2s+2}$ we'll factor out e^{-2s} and use that

$\mathcal{L}^{-1}\{e^{-sc} \cdot \mathcal{L}\{f(t)\}\} = u_c(t)f(t-c)$

$\mathcal{L}\{y\} = \frac{1+\frac{e^{-2s}}{s}}{s^2+2s+2} = \frac{1}{s^2+2s+2} + \frac{\frac{e^{-2s}}{s}}{s^2+2s+2} = \frac{1}{s^2+2s+2} + e^{-2s} \cdot \frac{1}{s(s^2+2s+2)}$

Solving a discontinuous Differential Equation

Example: Solve the Initial Value Problem

$$y'' + 2y' + 2y = u_2(t) \quad \text{with } y(0) = 0 \text{ and } y'(0) = 1$$

We need to find the Inverse Laplace of:

$\mathcal{L}\{y\} = \frac{1}{s^2+2s+2} + e^{-2s} \cdot \frac{1}{s(s^2+2s+2)}$

We earlier found the Laplace Inverse of $\frac{1}{s^2+2s+2}$ is: $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} = e^{-t}sin(t)$

To find the Laplace Inverse of $e^{-2s} \cdot \frac{1}{s(s^2+2s+2)}$ we'll use that

$\mathcal{L}^{-1}\{e^{-sc} \cdot \mathcal{L}\{f(t)\}\} = u_c(t)f(t-c)$ where $\mathcal{L}\{f(t)\} = \frac{1}{s(s^2+2s+2)}$

So, it remains to find:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/2}{s} + \frac{(-1/2)s-1}{s^2+2s+2}\right\} = \underbrace{\mathcal{L}^{-1}\left\{\frac{1/2}{s}\right\}}_{\frac{1}{2}} + \underbrace{\mathcal{L}^{-1}\left\{\frac{(-1/2)s-1}{s^2+2s+2}\right\}}_{e^{-t}\left(\frac{-1}{2}cos(t)-\frac{1}{2}sin(t)\right)}$$

Splitting $\frac{1}{s(s^2+2s+2)}$ using Partial Fractions: $\frac{A}{s} + \frac{Bs+C}{s^2+2s+2} = \frac{1/2}{s} + \frac{(-1/2)s-1}{s^2+2s+2}$

Factoring out $\frac{1}{2}$ we find: $\mathcal{L}^{-1}\left\{\frac{1/2}{s}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = \frac{1}{2}$

Vertex: $\frac{(-1/2)s-1}{s^2+2s+2} = \frac{(-1/2)s-1}{(s+1)^2+1} = \frac{(-1/2)(s+1)-1/2}{(s+1)^2+1} = \mathcal{L}\left\{e^{-t}\left(\frac{-1}{2}cos(t) - \frac{1}{2}sin(t)\right)\right\}$

$s+1$ signals us to use: $\mathcal{L}\{e^{ct} \cdot g(t)\} = G(s-c)$ where $G(s) = \mathcal{L}\{g(t)\}$

We need the entire function in $s+1$, so we'll rewrite the top

$y = e^{-t}sin(t) + u_2(t) \cdot \left[\frac{1}{2} + e^{-(t-2)} \cdot \left(\frac{-1}{2}cos(t-2) - \frac{1}{2}sin(t-2)\right)\right]$

Solving a discontinuous Step Functions

Example: Solve the Initial Value Problem

$$y' + 4y = u_2(t) \cdot sin(3t-6) \quad \text{with } y(0) = 0$$

Solution: **Taking the Laplace Transform of both sides**, we get:

$$\underbrace{\mathcal{L}\{y'\}} + 4\mathcal{L}\{y\} = \mathcal{L}\{y' + 4y\} = \mathcal{L}\{u_2(t) \cdot sin(3t-6)\} = e^{-2s} \cdot \frac{3}{s^2+9}$$

On the right hand side we'll to use that: $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-sc} \cdot \mathcal{L}\{f(t)\}$

Since $c = 2$, we rewrite $sin(3t-6) = sin(3(t-2))$ and compute:

$\mathcal{L}\{u_2(t) \cdot sin(3t-6)\} = e^{-2s} \cdot \frac{3}{s^2+3^2}$

Cleaning this equation up, we have: $\underbrace{(s \cdot \mathcal{L}\{y\} - 0)} + 4\mathcal{L}\{y\} = e^{-2s} \cdot \dfrac{3}{s^2+9}$

To solve for $\mathcal{L}\{y\}$, we collect all terms involving $\mathcal{L}\{y\}$ on one side to get:

$$(s+4) \cdot \mathcal{L}\{y\} = e^{-2s} \cdot \frac{3}{s^2+9} \Rightarrow \mathcal{L}\{y\} = e^{-2s} \cdot \frac{3}{(s+4)(s^2+9)}$$

We need to find the Laplace Inverse, $y = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\}$, for the above $\mathcal{L}\{y\}$

To find the Laplace Inverse we'll use: $\mathcal{L}^{-1}\{e^{-sc} \cdot \mathcal{L}\{f(t)\}\} = u_c(t) f(t-c)$

We'll use partial fractions to split: $\mathcal{L}\{f(t)\} = \frac{3}{(s+4)(s^2+9)} = \frac{\frac{3}{25}}{s+4} + \frac{\frac{-3}{25}s + \frac{12}{25}}{s^2+9}$

$$y = \mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{\frac{3}{25}}{s+4} + \frac{\frac{-3}{25}s+\frac{12}{25}}{s^2+9}\right)\right\} = \mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{3}{25}\frac{1}{s+4} - \frac{3}{25}\frac{s}{s^2+3^2} + \frac{4}{25}\frac{3}{s^2+3^2}\right)\right\}$$

$$= \mathcal{L}^{-1}\left\{e^{-2s}\mathcal{L}\left\{\frac{3}{25}e^{-4t} - \frac{3}{25}cos(3t) + \frac{4}{25}sin(3t)\right\}\right\}$$

Thus: $y = u_2(t) \cdot \left(\frac{3}{25}e^{-4(t-2)} - \frac{3}{25}cos\left(3(t-2)\right) + \frac{4}{25}sin\left(3(t-2)\right)\right)$

Definition of Dirac Delta Function

We looked at functions with discontinuities when we studied step functions.

Step functions are good for modeling a process that suddenly "turns on", like an electrical switch.

However, in some scenarios a large force can be applied in a short time, like an electrical surge.

To model this, we define a function $g(t)$ that is over a short interval centered around some time $t = 0$: $-\tau < t < \tau$ for a small $\tau > 0$. This function depends on τ, so we'll write $g(t) = d_\tau(t)$.

$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & \text{otherwise} \end{cases}$$

The total impulse, $I(\tau)$, of the force of $g(t)$ over the interval $-\tau < t < \tau$ can be found as the integral: $I(\tau) = \int_{-\tau}^{\tau} g(t)dt$

Notice that, with this definition, $I(\tau) = 1$ for every τ:

$$I(\tau) = \int_{-\tau}^{\tau} g(t)dt = \int_{-\tau}^{\tau} \frac{1}{2\tau}dt = \frac{1}{2\tau}\int_{-\tau}^{\tau} dt = \frac{1}{2\tau} \cdot 2\tau = 1$$

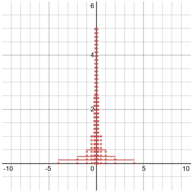
Definition of Dirac Delta Function

$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & \text{otherwise} \end{cases}$

$I(\tau) = \int_{-\tau}^{\tau} g(t)dt = 1$ for every τ

In a theoretical impulse function, we want to consider this force occurring at a single moment in time. That is, by letting $\tau \to 0$

As $\tau \to 0$, the function $g(t) = d_\tau(t)$ has a larger spike over a shorter period of time, since the impulse, $I(\tau)$, is fixed at $I(\tau) = 1$ for any τ.



We define the unit impulse function, $\delta(t)$, to behave like the limit: $\lim_{\tau \to 0} d_\tau(t)$

In particular, we define $\delta(t)$ to be a function with the properties:

$$\delta(t) = 0, \text{ for } t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Definition of Dirac Delta Function

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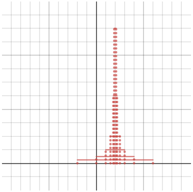
$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

We can generalize the above impulse function to spike at any time $t = t_o$ by evaluating at $(t - t_o)$, effectivley shifting the spike to time $t = t_o$.
This generalized version is called the Dirac Delta Function:

$$\delta(t - t_o) = 0, \text{ for } t \neq t_o$$

$$\int_{-\infty}^{\infty} \delta(t - t_o) dt = 1$$



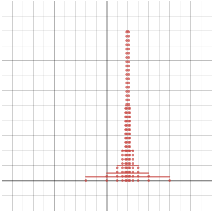
Integration of the Dirac Delta Function

We defined the generalized Dirac Delta Function as:

$$\delta(t - t_o) = 0, \text{ for } t \neq t_o \qquad \int_{-\infty}^{\infty} \delta(t - t_o)dt = 1$$

We can visualize $\delta(t - t_o)$ as the limit as $\tau \to 0$ of:

$$d_\tau(t - t_o) = \begin{cases} \frac{1}{2\tau} & t_o - \tau < t < t_o + \tau \\ 0 & \text{otherwise} \end{cases}$$



For a function, $f(t)$, we will compute:

$$\int_{-\infty}^{\infty} \delta(t - t_o) \cdot f(t)dt = \lim_{\tau \to 0^+} \int_{-\infty}^{\infty} d_\tau(t - t_o) \cdot f(t)dt$$

$$= \lim_{\tau \to 0^+} \int_{t_o - \tau}^{t_o + \tau} d_\tau(t - t_o) \cdot f(t)dt = \lim_{\tau \to 0^+} \frac{1}{2\tau} \int_{t_o - \tau}^{t_o + \tau} f(t)dt$$

$$= \lim_{\tau \to 0^+} \cancel{\frac{1}{2\tau} 2\tau} \cdot f(t^*) = \lim_{\tau \to 0^+} f(t^*) = f(t_o)$$

By the Mean Value Theorem, if $F(t)$ is an antiderivative of $f(t)$ then $\exists t^*$ with $t_o - \tau < t^* < t_o + \tau$ so that $f(t^*) = \frac{F(t_o + \tau) - F(t_o - \tau)}{2\tau}$

Thus, we have that: $\int_{t_o - \tau}^{t_o + \tau} f(t)dt = 2\tau \cdot f(t^*)$

Since $t_o - \tau < t^* < t_o + \tau$, we have that $\lim_{\tau \to 0} t^* = t_o$

Laplace Transform of the Dirac Delta Function

We saw that for a function $f(t)$:

$$\int_{-\infty}^{\infty} \delta(t - t_o) \cdot f(t)dt = f(t_o)$$

We will use this to compute the Laplace transform of $\delta(t - t_o)$:

$$\mathcal{L}\left\{\delta(t - t_o)\right\} = \int_0^{\infty} e^{-st} \cdot \delta(t - t_o)dt$$

$$= \int_0^{\infty} e^{-st} \cdot \delta(t - t_o)dt + \underbrace{\int_{-\infty}^{0} e^{-st} \cdot \delta(t - t_o)dt}_{=0}$$

$$= \int_{-\infty}^{\infty} e^{-st} \cdot \delta(t - t_o)dt = e^{-st_o}$$

Recall that for $t \neq t_o$, $\delta(t - t_o) = 0$

Thus for $t_o > 0$, we have that $\underbrace{\int_{-\infty}^{0} e^{-st} \cdot \delta(t - t_o)dt}= 0$

So, we can conclude that: $\mathcal{L}\left\{\delta(t - t_o)\right\} = e^{-st_o}$

Solving a discontinuous Differential Equation

Example: Solve the Initial Value Problem

$$y'' + 6y' + 25y = \delta(t-2) \quad \text{with } y(0) = 0 \text{ and } y'(0) = 0$$

Solution: **Taking the Laplace Transform of both sides**, we get:

$$\underbrace{\mathcal{L}\{y''\}}_{s^2\cdot\mathcal{L}\{y\}-s\cdot 0-0} + 6\,\underbrace{\mathcal{L}\{y'\}}_{s\cdot\mathcal{L}\{y\}-0} + 25\mathcal{L}\{y\} = \mathcal{L}\{y'' + 6y' + 25y\} = \mathcal{L}\{\delta(t-2)\} = e^{-2s}$$

Cleaning this equation up, we have: $\underbrace{s^2\mathcal{L}\{y\}} + 6\,\underbrace{s\mathcal{L}\{y\}} + 25\mathcal{L}\{y\} = e^{-2s}$

To solve for $\mathcal{L}\{y\}$, we factor our $\mathcal{L}\{y\}$ to get: $\left(s^2 + 6s + 25\right)\cdot\mathcal{L}\{y\} = e^{-2s}$

We need to find the Laplace Inverse, $y = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\}$, for $\mathcal{L}\{y\}$ where:

$$\mathcal{L}\{y\} = e^{-2s}\cdot\frac{1}{s^2 + 6s + 25} = e^{-2s}\cdot\frac{1}{(s+3)^2 + 16} = e^{-2s}\cdot\frac{1}{(s+3)^2 + 4^2}$$

$$= e^{-2s}\cdot\frac{1}{4}\frac{4}{(s+3)^2 + 4^2} = \frac{1}{4}e^{-2s}\cdot\mathcal{L}\left\{e^{-3t}sin(4t)\right\}$$

To find the Laplace Inverse we use: $\mathcal{L}^{-1}\{e^{-sc}\cdot\mathcal{L}\{f(t)\}\} = u_c(t)f(t-c)$

To find $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+25}\right\}$ we will put $s^2 + 6s + 25$ in vertex form since it is not factorable: $s^2 + 6s + 25 = (s+3)^2 + 16$

Now that it is in vertex, we'll use a shifted $\mathcal{L}\{sin(\alpha t)\} = \frac{\alpha}{s^2+\alpha^2}$ with $\alpha = 4$

So, we can conclude: $y = \frac{1}{4}u_2(t)\left[e^{-3(t-2)}sin(4(t-2))\right]$

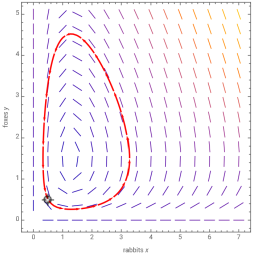
Introduction to Systems of Diff. Eq.

In some applications, we may need our model to track multiple variables that depend on a single independent variable.
We have discussed population models of a species.
We could have two species whose populations change over time, but are impacted by each other.
The most common application of this is a Predator-Prey model where one species relies on the other for food.
Consider the population of rabbits and foxes, represented by $x(t)$ and $y(t)$, respectively.

For positive values a, c, α, γ

$$\frac{dx}{dt} = ax - \alpha xy$$

$$\frac{dy}{dt} = -cx + \gamma xy$$



In this model, the fox population benefits from fox-rabbit interactions (the xy term) with a positive constant γ in its differential equation. Conversely, the rabbit population is negatively impacted by these interactions, so there is a negative constant $-\alpha$ of the interactions in the differential equation for x.

Introduction to Systems of Diff. Eq.

In general, for a set of n functions $x_1(t), \dots, x_n(t)$, a System of Differential Equations for x_i's is given by:

$$\begin{aligned} x_1'(t) &= F_1(t, x_1, x_2, \dots, x_n) \\ x_2'(t) &= F_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ x_n'(t) &= F_n(t, x_1, x_2, \dots, x_n) \end{aligned}$$

Notice that each derivative x_i' depends on the independent variable t as well as the other dependent variables, x_j for $1 \leq j \leq n$ and thus is written as a function of them: $x_i'(t) = F_i(t, x_1, x_2, \dots, x_n)$

Similar to a solution of a single diff. eq. being a function that made the diff. eq. true, a solution to a system of differential equations is a set of functions:
$x_1 = \phi_1(t), x_2 = \phi_2(t), \dots, x_n = \phi_n(t)$
that make ALL of the above differential equations true.

An initial value problem is a system of differential equations together with initial conditions for each x_i at a value $t = t_o$.

Theorem: For a set of n function $F_1, \dots, F_n$ with n^2 partial derivatives $\frac{\delta F_1}{\delta x_1}, \cdots, \frac{\delta F_1}{\delta x_n}, \cdots, \frac{\delta F_n}{\delta x_1}, \cdots, \frac{\delta F_n}{\delta x_n}$ that are continuous on a region R then there is a t-interval $t_o - h < t < t_o + h$ on which there exists a unique solution to a system of differential equations with initial conditions.

Linear vs.Non-Linear Systems of Diff. Eq.

For single differential equations of the form: $y'(t) = F(t, y)$
We called a diff. eq. *linear* if it could be written as:

$$y'(t) = p(t) \cdot y + g(t)$$

Similarly, we call a system of diff. eq.'s *linear* if it can be written in the form:

$$\begin{aligned} x_1'(t) &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2'(t) &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_n'(t) &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned}$$

Ex: The system of differential equations:

$$\begin{aligned} x_1'(t) &= t^2x_1 + (3-t)x_2 + ln(t)x_3 + 6t \\ x_2'(t) &= cos(4t)x_1 + 2tx_2 + t^3x_3 - 5 \\ x_3'(t) &= sin(t)x_1 + \tfrac{1}{t}x_2 + \sqrt{t}x_3 \end{aligned}$$

is a linear system of differential equations.
Non-Ex: The predator-prey model:

$$\begin{aligned} \tfrac{dx}{dt} &= ax - \alpha xy \\ \tfrac{dy}{dt} &= -cx + \gamma xy \end{aligned}$$

is not a linear system of differential equations due to the xy terms.
Also similar to our definition for single differential equations, we call a linear system *homogeneous* if $g_i(t) = 0$ for all i and all values of t.
We call a system of differential equations *nonhomogeneous* otherwise.

Higher Order Diff. Eq. and Systems of Diff. Eq.

Recall that we define the *order* of a differential equation as the highest derivative that arises in the differential equation.

Consider the spring-mass application that can model the position, $u(t)$, of a mass on a spring by the second order differential equation:

$$mu'' + \gamma u' + ku = F(t)$$

We can rewrite this second order differential equation as a system of differential equations if we consider both the position, $u(t)$, and the velocity, $v(t) = u'(t)$

With this , we have: $u''(t) = v'(t)$, and we can re-write our second order differential equation as:

$$mv'(t) + \gamma v(t) + ku = g(t)$$

Which can be solved for $v'(t)$ as: $v'(t) = -\frac{\gamma}{m} v(t) - \frac{k}{m} u(t) + \frac{1}{m} g(t)$

This equation, together with the (simpler) differential equation $u'(t) = v(t)$ gives the system of differential equations:

$$u'(t) = v(t)$$
$$v'(t) = -\frac{\gamma}{m} v(t) - \frac{k}{m} u(t) + g(t)$$

In general, we can change an order n differential equation given by:

$$y^{(n)}(t) = F\left(t, y, y', \ldots, y^{(n-1)}\right)$$

into an n-dimensional system of differential equations by making the change:

$$x_1 = y,\ x_2 = y',\ \ldots,\ x_n = y^{(n-1)}$$

which gives the system:

$$x_1' = x_2,\ x_2' = x_3,\ \ldots,\ x_{n-1}' = x_n,\ x_n' = F\left(t, x_1, x_2, \ldots, x_n\right)$$

Converting Systems of Equations to Second-Order Diff. Eq.

We saw that a second-order differential equation can be converted into a 2x2 systems of first order differential equations.

Similarly, we can convert a 2x2 system of diff. eq. with constant coefficients of the form:

$$x' = a \cdot x + b \cdot y$$
$$y' = c \cdot x + d \cdot y$$

into a second-order diff. eq. with constant coefficients.

Since we know how to solve such second-order diff. eq., this will give us a method to solve such systems of equations.

It's important to note, that while this does give us a method to solve systems of equations, it will not be our main method for solving them.

Our main method to solve such systems will involve the use of matrices, and is preferable because it will give us a better graphical understanding of solutions and will extend more efficiently to higher dimensional systems.

Converting Systems of Equations to Second-Order Diff. Eq.

Consider the 2x2 system of differential equations:

$$\begin{aligned} x' &= a \cdot x + b \cdot y \\ y' &= c \cdot x + d \cdot y \Rightarrow c \cdot x = y' - d \cdot y \end{aligned}$$

To convert this to a second order differential equation of y, we can compute the derivative of both sides of the second equation to get:

$$y'' = c \cdot x' + d \cdot y'$$

Replacing $x' = a \cdot x + b \cdot y$, from the first eq, in this equation for y'' gives:

$$\begin{aligned} y'' &= c \cdot x' + d \cdot y' \\ &= c \cdot (ax + by) + d \cdot y' \\ &= a\,(y' - d \cdot y) + cb \cdot y + d \cdot y' \\ &= a \cdot y' - ad \cdot y + cb \cdot y + d \cdot y' \\ &= (a + d) \cdot y' + (bc - ad) \cdot y \end{aligned}$$

We now have our 2x2 system written as a second order differential equation.
We can write this in our standard form by moving all terms to one side:

$$y'' - (a + d) \cdot y' + (ad - bc) \cdot y = 0$$

As a second-order diff. eq. we can solve for $y(t)$
Once $y(t)$ is known, you can solve for x using $c \cdot x = y' - d \cdot y$
Note: If $c = 0$ then you cannot solve the above equation for x and will need to solve for x in the first equation, as a first-order linear diff. eq.
This will give us solutions for both functions, $x(t)$ and $y(t)$, in the system.

Converting Systems of Equations to Second-Order Diff. Eq.

Solve the system of differential equations:

$$\begin{aligned} x' &= -3x + 6y \\ y' &= x + 2y \end{aligned}$$

by changing it into a second-order diff. eq. with constant coefficients.

Solution: Taking the derivative of both sides of the second equation gives:

$$y'' = x' + 2y'$$

Replacing x' in this equation with $x' = -3x + 6y$ from the first equation gives:

$$\begin{aligned} y'' &= -3x + 6y + 2y' \\ &= -3(y' - 2y) + 6y + 2y' \\ &= -3y' + 6y + 6y + 2y' = -y' + 12y \end{aligned}$$

To get this equation completely in terms of y and its derivatives, we can use that $x = y' - 2y$ from the second equation in the system.

Writing this in our standard form for second-order diff. eq. yields:

$$y'' + y' - 12y = 0$$

We solved this differential equation earlier to find:

$$y = c_1 e^{-4t} + c_2 e^{3t}$$

To find $x = y' - 2y$ we will need to compute $y' = -4c_1 e^{-4t} + 3c_2 e^{3t}$

Thus: $$x = \underbrace{-4c_1 e^{-4t} + 3c_2 e^{3t}}_{y'} - 2\underbrace{\left(c_1 e^{-4t} + c_2 e^{3t}\right)}_{y} = -6c_1 e^{-4t} + c_2 e^{3t}$$

Note: Since c_1 and c_2 are defined as constants in the solution for $y(t)$, we do not want to replace $-6c_1$ in the equation for $x(t)$, as we will lose this relationship between the constant coefficients in $y(t)$ and $x(t)$

Systems of Linear Diff. Eq. with Matrices

Recall that a system of diff. eq. of two functions is linear if it can be written as:

$$x' = p_{11}(t) \cdot x + p_{12}(t) \cdot y + g_1(t)$$
$$y' = p_{21}(t) \cdot x + p_{22}(t) \cdot y + g_2(t)$$

We will explore the homogeneous case where $g_1(t) = g_2(t) = 0$

We will restrict our study further to the case of constant coefficients.

That is, the case where $a = p_{11}(t)$ is constant, $b = p_{12}(t)$ is constant, ...

$$x' = a \cdot x + b \cdot y$$
$$y' = c \cdot x + d \cdot y$$

To represent the pair of functions, x and y, we will use a vector function:

$$\vec{Y} = \langle x, y \rangle = \begin{pmatrix} x \\ y \end{pmatrix}$$

And we will use a matrix, A, to represent the coefficients:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We will need to study properties of matrices before using them to solve systems of differential equations.

Operations with Matrices

We will need to learn some operations of matrices to proceed with our study of Linear Systems of Diff. Eq. with constant coefficients.

We will write a generic $n \times n$ matrix as: $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

We will call each value in the matrix an entry

We will call a horizontal set of entries a row

We will call a vertical set of entries a column

The indices of an entry a_{ij} give the row and column of the entry.

For example, the entry a_{21} is the entry in the 2^{nd} row and the 1^{st} column

We will sometimes write the entry in the i^{th} row and j^{th} column of the matrix A as $(A)_{ij}$

In general, the number of rows and columns of a matrix can be different sizes.

In our study of Linear Systems of Differential Equations, we will restrict our study to square matrices where the number of $^{\#}$ of rows $=$ $^{\#}$ of columns

Operations with Matrices

When we are *Adding* two matrices A and B, they must be the same size. That is, A and B must have the same number of rows and columns.

To *Add* two matrices, we add entries that are in the same location

$$A + B = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} + \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}}_{B}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nn}+b_{nn} \end{pmatrix}$$

Notice that the entry in the i^{th} row and j^{th} column of the matrix $A + B$, which we write as $(A + B)_{ij}$, is given by:

$$(A + B)_{ij} = A_{ij} + B_{ij}$$

Example: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 8 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ 9 & 2 \end{pmatrix}$

Operations with Matrices

Multiplication with Matrices depends on what we multiply the Matrix with

We can multiply a Matrix with a scalar, a vector, or another matrix.

We will start with Multiplication of a Matrix with a scalar.

For the constant scalar c and the matrix A, we define multiplication:

$$c \cdot A = c \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} c \cdot a_{11} & c \cdot a_{12} & \cdots & c \cdot a_{1n} \\ c \cdot a_{21} & c \cdot a_{22} & \cdots & c \cdot a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c \cdot a_{n1} & c \cdot a_{n2} & \cdots & c \cdot a_{nn} \end{pmatrix}$$

We multiply each entry of (A) by c

Operations with Matrices

Multiplication with Matrices depends on what we multiply the Matrix with

For a Matrix, A, and a vector $\vec{v}$ we define multiplication:

$$A\vec{v} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} \cdot v_1 + a_{12} \cdot v_2 + \cdots + a_{1n} \cdot v_n \\ a_{21} \cdot v_1 + a_{22} \cdot v_2 + \cdots + a_{2n} \cdot v_n \\ \vdots \\ a_{n1} \cdot v_1 + a_{n2} \cdot v_2 + \cdots + a_{nn} \cdot v_n \end{pmatrix}$$

The result of this multiplication will be another vector.

The entry in the i^{th} row of the new vector will be found by taking the dot product of the i^{th} row of A with $\vec{v}$.

Example: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 + 2 \cdot 6 \\ 3 \cdot 3 + 4 \cdot 6 \end{pmatrix} = \begin{pmatrix} 3 + 12 \\ 9 + 24 \end{pmatrix} = \begin{pmatrix} 15 \\ 33 \end{pmatrix}$

Note: To be well defined, the $^{\#}$ of columns of A must equal the $^{\#}$ of rows of $\vec{v}$

Operations with Matrices

Multiplication with Matrices depends on what we multiply the Matrix with
For a Matrix, A, and another Matrix, B, we define multiplication:

$$A \cdot B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} (A \cdot B)_{11} & \cdots & \\ \vdots & \ddots & \\ & & \end{pmatrix}$$

The result of this multiplication will be another matrix.
The entry in the i^{th} row and j^{th} column of the new matrix will be found by taking the dot product of the i^{th} row of A with the j^{th} column of B.
That is, ij-entry of $A \cdot B$ is given by:

$$(A \cdot B)_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots a_{in} \cdot b_{nj}$$

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 8 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 + 2 \cdot 6 & 1 \cdot 8 + 2 \cdot (-2) \\ 3 \cdot 3 + 4 \cdot 6 & 3 \cdot 8 + 4 \cdot (-2) \end{pmatrix} = \begin{pmatrix} 15 & 4 \\ 33 & 16 \end{pmatrix}$$

Note: To be well defined, the # of columns of A must equal the # of rows of B

Operations with Matrices

Note 1: Unlike with numbers, the order in which we multiply Matrices matters.
For two numbers, a and b, we have that $a \cdot b = b \cdot a$.
However, in general, for two matrices, A and B, we have that $A \cdot B \neq B \cdot A$
Though, there are exceptions to this.
Note 2: There is a number, 1, that can be multiplied by any number and the number doesn't change.
In other words, $1 \cdot x = x = x \cdot 1$ for any number, x
Similarly, there is an $n \times n$ matrix, I, that can be multiplied by any $n \times n$ matrix and the matrix doesn't change.
In other words, $I \cdot M = M = M \cdot I$ for any matrix, M
This matrix, I, is called the identity matrix and is the matrix with 1's on the diagonal and 0's in all other entries:

$$I = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Note 3: Multiplications by a scalar c is equivalent to multiplying by $c \cdot I$.
For a vector, $\vec{v}$: $c \cdot \vec{v} = (c \cdot I)\vec{v}$ and for a matrix, A: $c \cdot A = (c \cdot I)A$

Self-Operations with Matrices

We just discussed the operations of adding two matrices and multiplying a matrix with either another matrix, a vector, or a scalar.

Some operations of matrices do not involve a second value.

We have similar functions for scalar numbers such as negation and complex conjugation.

Negation is an operations on a single value, instead of two: e.g. $-(-3) = 3$

Complex conjugation is, also, for a single complex value: e.g. $\overline{3+i} = 3 - i$

There are some useful Self-operations on Matrices, as well.

For a matrix, A, we define $\overline{A}$ as the matrix such that each entry of $\overline{A}$ is the complex conjugate of the corresponding entry of A.

Example: $\overline{\begin{pmatrix} 1+i & 3-2i \\ 3-i & 4+3i \end{pmatrix}} = \begin{pmatrix} \overline{1+i} & \overline{3-2i} \\ \overline{3-i} & \overline{4+3i} \end{pmatrix} = \begin{pmatrix} 1-i & 3+2i \\ 3+i & 4-3i \end{pmatrix}$

Self-Operations with Matrices

Another self operation of a matrix, A, is called the transposition, we denote A^T

The entries of the transposed matrix, A^T, can be found by changing the positions of the entries so that:

$$(A)_{ij} = (A^T)_{ji}$$

That is, the entry in the i^{th} row and j^{th} column of A becomes the entry in the j^{th} row and i^{th} column of A^T

We can also think of this as moving the rows of A to the columns of A^T

We can, similarly, think of this as moving the columns of A to the rows of A^T

Example: For the Matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ we can compute the transpose as:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Inverse and Determinant of a Matrices

For scalar numbers, our four basic functions are addition, subtraction, multiplication, and division.

We can understand subtraction as adding the negative: e.g. $6 - 3 = 6 + (-3)$

We can define subtraction of matrices, similarly, as: $A - B = A + (-1)B$

Since we understand scalar multiplication, here by (-1), and addition of matrices, we can understand how to subtract matrices.

We can, also, understand division as multiplying by the inverse: e.g. $6 \div 3 = 6 \cdot \frac{1}{3}$

For a number, a, the inverse of a number, written a^{-1}, is defined so that $a^{-1} \cdot a = 1 = a \cdot a^{-1}$

Similarly, for an $n \times n$ matrix, A, the inverse of a matrix, written A^{-1}, is defined so that $A^{-1} \cdot A = I = A \cdot A^{-1}$

Note: for scalar numbers, we can take the inverse of any number except zero.

In the context of matrices, there will matrices that do not have an inverse.

We will call a matrix that has an inverse *invertible* or *non-singular*, and call a matrix without an inverse either *non-invertible* or *singular*.

Inverse and Determinant of a Matrices

Since some matrices are invertible and some are not, it will be helpful to have a way of determining this.

The *determinant* of a matrix, written $det(A)$, gives us a way of telling whether or not a matrix, A, is invertible

The *determinant* of a matrix is defined for any $n \times n$ matrix, though for our course we will restrict our study to 2×2 matrices.

The determinant of the matrix can be found computationally:

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot d - b \cdot c$$

As the product: (the values on the diagonal) − (the values on the off-diagonal)

There is a lot of important theory about the determinant, many of which you'd learn in a Linear Algebra course.

The result that will be most important for our study is that:

$$det(A) \neq 0 \Leftrightarrow A \text{ is invertible}$$

So, this gives us a quick way to determine if a matrix is invertible.

Example: Compute the determinant:

$$det \begin{pmatrix} 2 & 1 \\ 6 & -3 \end{pmatrix} = 2 \cdot (-3) - 1 \cdot 6 = -6 - 6 = -12$$

Return to using Matrices with Linear Systesm of Diff. Eq.'s

We motivated our conversation about Matrices as a way to understand Linear Systems of differential equations.

As we said earlier, for this course we will focus on homogeneous Linear Systems with constant coefficients, which have the form:

$$\begin{aligned} x' &= a \cdot x + b \cdot y \\ y' &= c \cdot x + d \cdot y \end{aligned}$$

We will now use our understanding of matrices to recast these systems of differential equations.

Let $\vec{Y} = < x, y > = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Computing the multiplication of A on $\vec{Y}$ we get:

$$A \cdot \vec{Y} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \cdot x + b \cdot y \\ c \cdot x + d \cdot y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \vec{Y}'$$

So, we can rewrite the system of differential equations:

$$\begin{aligned} x' &= a \cdot x + b \cdot y \\ y' &= c \cdot x + d \cdot y \end{aligned}$$

as $\vec{Y}' = A\vec{Y}$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$

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Solutions of a Linear System

Consider the Linear System of Differential Equations:

$$\begin{aligned} x' &= 5x - 2y \\ y' &= -x + 4y \end{aligned} \qquad \leftrightarrow \vec{Y}' = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix} \vec{Y} \text{ where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

At this point, we don't have the tools to solve this system.

When we were studied first-order differential equations, we used direction fields to understand solutions of diff eq whose solutions we could not find.

Here, we will look at the direction field in the xy-plane, called the *phase plane*.

The direction field will show the direction of vectors tangent to solutions $\vec{Y}$

That is, directions will be given by $\vec{Y}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ scaled down to length 1.

Since we have formulas for $\vec{Y}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ in terms of x and y, we can compute directions of tangent vectors of $\vec{Y}$ at any point (x, y) in the phase plane.

So, it is easy to generate a direction field for solutions $\vec{Y}$ even when we cannot find solutions for $\vec{Y}$

Solutions of a Linear System

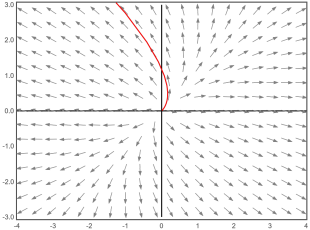
Consider the Linear System of Differential Equations:

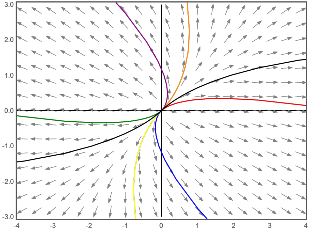
$$\begin{aligned} x' &= 5x - 2y \\ y' &= -x + 4y \end{aligned} \quad \leftrightarrow \vec{Y}' = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix} \vec{Y} \text{ where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

At the point $(x, y) = (1, 2)$ the tangent vector is given by:

$$\vec{Y}' = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5-4 \\ -1+8 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

This can be scaled down to length 1 by dividing by it's magnitude. Since we need many direction vectors to get a good picture of how our solutions behave, these computations are best left to a computer.

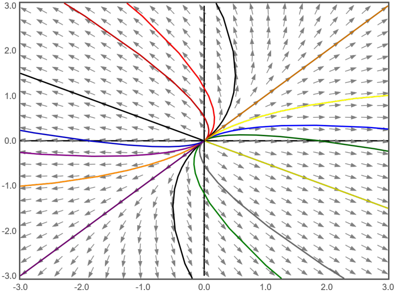




From a direction field, we can sketch solutions, $\vec{Y}$, tangent to the direction vec. A collection of graphs of solutions is called the *phase portrait*

Straight Line Solutions of a Linear System

We saw that we could sketch solutions to Linear Systems of differential equations given by $\vec{Y}' = A\vec{Y}$ using a direction field



Notice here, that we have some interesting solutions that stand out.
These solutions are straight lines, so we refer to them as *straight-line solutions*.
For straight line solutions, the tangent vector is parallel to the position vector.
So, $\vec{Y}'$ is a scalar mutliple of $\vec{Y}$
That is, for some scalar λ we have: $A\vec{Y} = \vec{Y}' = \lambda\vec{Y}$
So, these straight line solutions satisfy: $A\vec{Y} = \lambda\vec{Y}$
These straight line solutions will play an important role in finding all solutions.
To find these straight-line solutions, we need more from Linear Algebra.

Solutions of Linear Systems of Equations

We saw that that straight-line solutions, such that $\vec{Y}' = \lambda \vec{Y}$ for a constant λ, to Linear Systems of differential equations, given by $\vec{Y}' = A \cdot \vec{Y}$, satisfy:

$$A \cdot \vec{Y} = \lambda \vec{Y}$$

For us to find these straight-line solutions, $\vec{Y}$, we first need to understand how to solve Linear equations of the form:

$$A\vec{x} = \vec{b}$$

for an unknown vector $\vec{x}$ and known vector $\vec{b}$ with a matrix A

It can be shown that if A is invertible then for any vector $\vec{b}$ the equation $A \cdot \vec{x} = \vec{b}$ has a unique solution for $\vec{x}$

The solution can be found using the inverse, A^{-1}, as: $\vec{x} = A^{-1} \cdot \vec{b}$

We will be most concerned with such equations where $\vec{b} = \vec{0}$, consisting with 0's in all entries: $A \cdot \vec{x} = \vec{0}$

In this case where $A\vec{x} = \vec{0}$ and A is invertible, then $\vec{x} = A^{-1} \cdot \vec{0} = \vec{0}$

Thus, if A is invertible then the only solution of $A \cdot \vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

We, also, found that A is invertible $\Leftrightarrow det(A) \neq 0$

Conclusion: If $det(A) \neq 0$ then the only solution of $A \cdot \vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

Eigenvalues and Vectors

We saw that that straight-line solutions, such that $\vec{Y}' = \lambda \vec{Y}$ for a constant λ, to Linear Systems of differential equations, given by $\vec{Y}' = A \cdot \vec{Y}$, satisfy:

$$A \cdot \vec{Y} = \lambda \vec{Y}$$

We, also, found that if $det(A) \neq 0$ then the only solution of $A \cdot \vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

We will start by giving a definition to constants, λ, and vectors, $\vec{v}$, that satisfy the above equation we found to be true for straight-line solutions: $A \cdot \vec{v} = \lambda \vec{v}$

Definition: We call λ an *eigenvalue* with *eigenvector* $\vec{v} \neq \vec{0}$ if $A \cdot \vec{v} = \lambda \vec{v}$

We can see that these eigenvalues and eigenvectors play an important role in straight-line solutions to our Linear Systems of Differential Equations.

So, we need to study how to find them.

In our equation $A \cdot \vec{v} = \lambda \vec{v}$, we have different types of multiplications.

On the left, we multiply a Matrix to a vec. and on the right a scalar to a vec.

To have the same type of mult. recall that we can replace $\lambda \vec{v} = (\lambda I)\vec{v}$

With this, we can rewrite our equation: $A \cdot \vec{v} = \lambda \vec{v}$ as $A \cdot \vec{v} = (\lambda I)\vec{v}$

Since we now have the same type of multiplication, we can rewrite as:

$$(A - \lambda I) \cdot \vec{v} = \vec{0}$$

But since $\vec{v} \neq \vec{0}$ by def. of an eigenvector, we must have that $det(A - \lambda I) = 0$

Conclusion: If λ is an eigenvalue of A then $det(A - \lambda I) = 0$

Eigenvalues and Vectors

Definition: We call λ an *eigenvalue* with *eigenvector* $\vec{v} \neq \vec{0}$ if $A \cdot \vec{v} = \lambda \vec{v}$
Conclusion: If λ is an eigenvalue of A then $det(A - \lambda I) = 0$

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ this means that:

$$(a - \lambda) \cdot (d - \lambda) - b \cdot c = det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = det(A - \lambda I) = 0$$

Distributing the left hand side of this we get:

$$\lambda^2 - \underbrace{(a + d)}_{Tr(A)}\lambda + \underbrace{(ad - bc)}_{det(A)} = 0$$

This gives us a quadratic equation to solve to find the eigenvalues λ
$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$ is called the *characteristic equation* of A
Note that, in the characteristic equation the constant term is the $det(A)$

λ's coefficient is negative the sum of the diagonal a+d of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

This sum of the diagonal $a + d$ is called the Trace of A and written $Tr(A)$
Using this language, we have that the characteristic equation of A is:

$$\lambda^2 - Tr(A) \cdot \lambda + det(A) = 0$$

For each value λ we find as a solution to the char. eq. we can find the corresponding eigenvector, $\vec{v}$, using the definition of eigenvalue/eigenvector

Eigenvalues and Vectors Example

Example: Find the eigenvalue(s) and eigenvector(s) of:

$$A = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix}$$

Solution: To start we'll find eigenvalues as the sol's to the characteristic eq:

$$0 = det \begin{pmatrix} 5-\lambda & -2 \\ -1 & 4-\lambda \end{pmatrix} = (5-\lambda)(4-\lambda) - (-2)(-1) = \lambda^2 - 9\lambda + 18$$

Factoring $\lambda^2 - 9\lambda + 18 = (\lambda - 3) \cdot (\lambda - 6)$ we get: $\lambda_1 = 3$, $\lambda_2 = 6$

For $\lambda_1 = 3$, $\lambda_2 = 6$ we need to compute an eigenvector for each

$\lambda_1 = 3$ is an eigenvalue, so for it's eigenvector, $\vec{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 5a - 2b \\ -1a + 4b \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 3 \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a \\ 3b \end{pmatrix}$$

We can find a relationship between a, b from: $5a - 2b = 3a \Rightarrow a = b$

$\lambda_2 = 6$ is an eigenvalue, so for it's eigenvector, $\vec{v}_2 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 5a - 2b \\ -1a + 4b \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 6 \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6a \\ 6b \end{pmatrix}$$

We can find a relationship between a, b from: $5a - 2b = 6a \Rightarrow a = -2b$

Linearly Independent Vectors

Our study of Linear Systems of Differential Equations has relied on theory from Linear Algebra.

We will need one more definition and result from Linear Algebra to find the General Solution to a system of differential equations.

Definition: A set of k vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are called *linearly dependent* if there exists a set of constants $c_1, c_2, \ldots, c_k$ with $c_i \neq 0$ for at least one $1 \leq i \leq k$ so that:

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots c_k\vec{x}_k = 0$$

If $c_i \neq 0$ we can solve the above equation for $\vec{x}_i$:

$$\vec{x}_i = -\frac{c_1}{c_i}\vec{x}_1 - \frac{c_2}{c_i}\vec{x}_2 - \cdots \frac{c_{i-1}}{c_i}\vec{x}_{i-1} - \frac{c_{i+1}}{c_i}\vec{x}_{i+1} \cdots \frac{c_k}{c_i}\vec{x}_k$$

So, we can also think of a set of vectors as being linearly dependent if one of the vectors can be written as a linear combination of the others.

A set of vectors that are not linearly dependent are called *linearly independent*

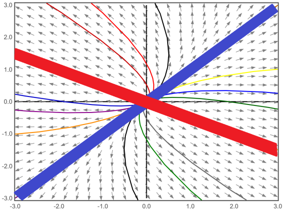
The main result that we will need is that if $\lambda_1, \cdots, \lambda_k$ are distinct eigenvalues of a matrix A then the associated eigenvectors $\vec{v}_1, \cdots, \vec{v}_k$ are linearly independent.

This linear independence will be needed to know when an infinite family of solutions forms the General Solution.

The General Solution of a Linear System of Diff. Eq.

Studying the direction field and phase portrait of the Linear System of Diff Eq:

$$\vec{Y}' = A\vec{Y} \text{ where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } A = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix}$$



we noticed two straight-line solutions.
While we have not found those straight-line solutions yet, we observed that along a straight-line solution $\vec{Y}' = \lambda\vec{Y}$ and thus:

$$A\vec{Y} = \vec{Y}' = \lambda\vec{Y}$$

This led us to the idea of eigenvalue, λ, with eigenvector, $\vec{v}$, for which $A\vec{v} = \lambda\vec{v}$
Moreover, for this A we found the eigenvalues/eigenvectors to be:

$\lambda_1 = 3, \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = 6, \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

But what about the other solutions we see in the phase portrait?
We will need two theorems to find the General Solution

The General Solution of a Linear System of Diff. Eq.

Theorem: Principle of Superposition

If $Y_1\vec{(}t)$ and $\vec{Y}_2(t)$ are solutions to the Linear System of Differential Equations:

$$\vec{Y}' = A\vec{Y}$$

then for any constants c_1 and c_2:

$$Y\vec{(}t) = c_1\vec{Y}_1(t) + c_2\vec{Y}_2(t)$$

is also a solution.

This can be quickly verified using linearity of the derivative and matrix mult.

So, we can build a collection of new solutions from ones we find.

But how do we know if the collection is all solutions, i.e. the General Solution?

Theorem: If $\vec{Y}_1(t), \ldots, \vec{Y}_n(t)$ are solutions to the Linear System of n Diff. Eq.:

$$\vec{Y}' = A\vec{Y}$$

and the set of vectors $\vec{Y}_1(t_o), \ldots, \vec{Y}_n(t_o)$ at a value $t = t_o$ are linearly independent then:

$$\vec{Y}(t) = c_1\vec{Y}_1(t) + \cdots + c_n\vec{Y}_n(t) \text{ forms the General Solution.}$$

In the case of 2-dimensional Linear Systems that we've been focusing on:

If we can find two solutions $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ that are linearly independent then:

$$\text{The General Solution is: } \vec{Y}(t) = c_1\vec{Y}_1(t) + c_2\vec{Y}_2(t)$$

Understanding the straight-line solutions will be the key to finding $\vec{Y}_1(t)$, $\vec{Y}_2(t)$

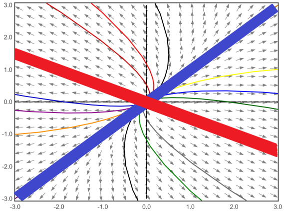
Note: If A has distinct eigenvalues, $\lambda_1 \neq \lambda_2$, the eigenvectors, and thus straight-line solutions, will be linearly independent.

Solutions of Systems of Diff. Eq.

We looked for solutions to Linear Systems of Diff Eq. in the example:

$$\vec{Y}' = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix} \cdot \vec{Y} \qquad \text{where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

On the direction field and phase portrait, we noticed straight-line solutions



We observed that straight-line sol's are scalar multiples of the tangent vector $\vec{Y}'$
That is, there are straight-line sol's, $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ so that $\vec{Y}_i'(t) = \lambda_i \vec{Y}_i(t)$
Moreover, each $Y_i(t)$ is a scalar multiple of an eigenvector, $\vec{v}_i$
As $\vec{Y}_i(t)$ varies with t, so does the scalar multiple, so we have: $\vec{Y}_i(t) = f_i(t) \cdot \vec{v}_i$
We, also, saw that, for a 2-dimensional systems like this one, if we can find two linearly independent solutions then the General Solution is given by:

$$\vec{Y} = c_1 \vec{Y}_1(t) + c_2 \vec{Y}_2(t)$$

Since A has distinct eigenvalues, and thus the eigenvectors are linearly ind., if we find these two straight-line sol's then we can build the Gen. Sol.
So, all that remains is to find these straight-line solutions.

Solutions of Systems of Diff. Eq.

For each $\vec{Y}_i(t)$ we know that $\vec{Y}_i(t) = f_i(t) \cdot \vec{v}_i$ and $\vec{Y}'(t) = \lambda \vec{Y}(t)$

We found e-vecs, $\vec{v}_i$, thus we need to find $f_i(t)$ to find $\vec{Y}_i(t)$

Recall from the beginning of the semester, for first order diff eq of the form:

$$y' = \lambda y$$

We found that $y = e^{\lambda t}$ is a solution.

From this insight, let's check if this is true for a straight-line solution, $\vec{Y}_i(t)$, to a linear system of diff. eq. that satisfies $\vec{Y}_i'(t) = \lambda_i \vec{Y}_i(t)$

That is, since straight-line sol $\vec{Y}_i' = \lambda_i \vec{Y}_i$ for an eigenvalue λ_i, let's check if:

$$\vec{Y}_i = \underbrace{e^{\lambda_i t}}_{f_i(t)} \vec{v}_i$$

is a solution to our linear system of diff. eq: $\vec{Y}_i{}' = A\vec{Y}_i$

So, we want to see if the differential equation is true for $\vec{Y}_i = e^{\lambda_i t} \vec{v}_i$:

Since $\vec{v}_i$ is an eigenvector with eigenvalue λ_i, we can rewrite $A\vec{v}_i = \lambda_i \vec{v}_i$

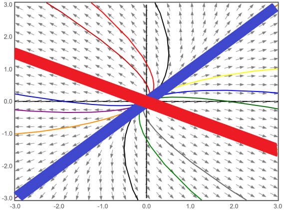
Since $\vec{v}_i$ is constant, $\left(e^{\lambda_i t} \vec{v}_i\right)' = \left(e^{\lambda_i t}\right)' \vec{v}_i$

So, we have shown that if A has an eigenvalue λ_i with eigenvector $\vec{v}_i$ then $\vec{Y}_i(t) = e^{\lambda_i t} \vec{v}_i$ is a solution to $\vec{Y}_i' = A\vec{Y}_i$

Solutions of Systems of Diff. Eq.

Returning to our example to find solutions to the Linear Systems of Diff Eq.:

$$\vec{Y}' = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix} \cdot \vec{Y} \qquad \text{where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$



For a 2-dimensional systems like this one, if we can find two linearly independent solutions then the General Solution is given by:

$$\vec{Y} = c_1 \vec{Y}_1(t) + c_2 \vec{Y}_2(t)$$

We now know that if A has an eigenvalue λ with eigenvector $\vec{v}$ then $\vec{Y}(t) = e^{\lambda t}\vec{v}$ is a solution to $\vec{Y}' = A\vec{Y}$

Since we found that $\lambda_1 = 3$ and $\lambda_2 = 6$ are eigenvalues with eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ we have: $\vec{Y}_1(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{Y}_2(t) = e^{6t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Thus, $\vec{Y}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{6t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is the general solution.

Componentwise Solutions of Systems of Diff. Eq.

Our example:

$$\vec{Y}' = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix} \cdot \vec{Y} \qquad \text{where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

can be written componentwise as:

$$\begin{aligned} x' &= 5x - 2y \\ y' &= -x + 4y \end{aligned}$$

We found the general solution:

$$\vec{Y}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

So, what are the scalar functions $x(t)$ and $y(t)$?

Notice that we can multiple each vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ by their scalar multiples to get:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{Y}(t) = \begin{pmatrix} c_1 e^{3t} \\ c_1 e^{3t} \end{pmatrix} + \begin{pmatrix} -2c_2 e^{6t} \\ c_2 e^{6t} \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} - 2c_2 e^{6t} \\ c_1 e^{3t} + c_2 e^{6t} \end{pmatrix}$$

And we can add these vectors componentwise.

Writing $\vec{Y}(t)$ as one vector, we get the solutions for x and y as the components of $\vec{Y}(t)$:

$$\begin{aligned} x &= c_1 e^{3t} - 2c_2 e^{6t} \\ y &= c_1 e^{3t} + c_2 e^{6t} \end{aligned}$$

Solutions of Systems of Diff. Eq. with Distinct Eigenvalues

We have now seen an example in which we found the General Solution to a Linear System of Differential Equations of the form:

$$\vec{Y}' = A \cdot \vec{Y} \qquad \text{where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let's recap that process in the case where A has distinct eigenvalues $\lambda_1 \neq \lambda_2$

Our general solution will depend on those eigenvalues and their eigenvectors.

To start, we find λ_1 and λ_2 as roots of the characteristic equation:

$$0 = det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc)$$

Separately for λ_1 and λ_2, we need to find eigenvectors $\vec{v}_1$ and $\vec{v}_2$ satisfying the definition of an eigenvalue/vector:

$$A\vec{v} = \lambda\vec{v}$$

For $\lambda_1 \neq \lambda_2$ the eigenvectors are linearly independent and thus:

$$\vec{Y} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 \text{ is the General Solution}$$

Note 1: The theory we covered changes the process of solving a system of diff eq into a Linear Algebra process of finding eigenvalues and eigenvectors.

Note 2: The solutions $\vec{Y}_1 = e^{\lambda_1 t}\vec{v}_1$ and $\vec{Y}_2 = e^{\lambda_2 t}\vec{v}_2$ correspond to the straight-line solutions in the phase plane.

Solutions of Systems of Diff. Eq. Example 2

Example: Find the General Solution of the Linear System of Diff. Eq. given by:

$$\vec{Y}' = \begin{pmatrix} 3 & -6 \\ -3 & 0 \end{pmatrix} \cdot \vec{Y} \qquad \text{where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

We start by finding the e-values λ_1 and λ_2 as roots of the characteristic eq:

$$0 = det \begin{pmatrix} 3 - \lambda & -6 \\ -3 & 0 - \lambda \end{pmatrix} = \lambda^2 - 3\lambda - 18$$

We get the values $\lambda_1 = -3$ and $\lambda_2 = 6$ by factoring this as: $0 = (\lambda + 3)(\lambda - 6)$
For $\lambda_1 = -3$, $\lambda_2 = 6$ we need to compute an eigenvector for each

$\lambda_1 = -3$ is an eigenvalue, so for it's eigenvector, $\vec{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 3a - 6b \\ -3a \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ -3 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = -3 \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -3a \\ -3b \end{pmatrix}$$

We can find a relationship between a, b from: $3a - 6b = -3a \Rightarrow a = b$

$\lambda_2 = 6$ is an eigenvalue, so for it's eigenvector, $\vec{v}_2 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 3a - 6b \\ -3a \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ -3 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 6 \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6a \\ 6b \end{pmatrix}$$

We can find a relationship between a, b from: $3a - 6b = 6a \Rightarrow a = -2b$

General Solution: $\vec{Y}(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Phase Portrait of Systems from Example 2

In Example 2, we found that the Genereal Solution to:

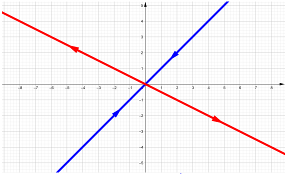
$$\vec{Y}' = \begin{pmatrix} 3 & -6 \\ -3 & 0 \end{pmatrix} \cdot \vec{Y} \qquad \text{where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

in terms of constants c_1 and c_2 are: $\vec{Y}(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

The Phase Portrait shows sol's of the System of Diff. Eq. in the $xy-$plane.
Let's sketch the Phase Portrait using the Gen. Sol. we found.
To start, we know that when $c_2 = 0$ the solution is: $\vec{Y}_1(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Similarly, when $c_1 = 0$ the solution is: $\vec{Y}_2(t) = c_2 e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Since these sol's give scalar multiples of the e-vecs, they give straight-line sol's

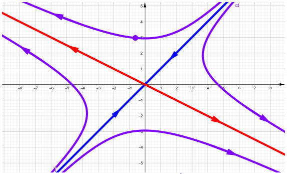


Along $\vec{Y}_1$, as $t \to \infty$, $e^{-3t} \to 0$ and thus $\vec{Y}_1$ goes to the origin
We indicate, with an arrow, the direction of the solution as t increases.
Along $\vec{Y}_2$, as $t \to \infty$, $e^{6t} \to \infty$ and thus $\vec{Y}_2$ becomes a larger multiple of $\vec{v}_2$

Phase Portrait of Systems from Example 2

For the General Solution: $\vec{Y}(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

The Phase Portrait is given by:



Along $\vec{Y}_1$, as $t \to \infty$, $e^{-3t} \to 0$ and thus $\vec{Y}_1$ goes to the origin
Along $\vec{Y}_2$, as $t \to \infty$, $e^{6t} \to \infty$ and thus $\vec{Y}_2$ becomes a larger multiple of $\vec{v}_2$
What about the curves of the other solutions, where $c_1 \neq 0$ and $c_2 \neq 0$?
Starting with some point on neither straight-line sol, what happens as $t \to \infty$?
As $t \to \infty$, we know that $\vec{Y}_1 \to \vec{0}$, thus $\vec{Y}(t)$ gets asymptotically close to $c_2 e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, a scalar multiple of $\vec{v}_2$

Conversely, as $t \to -\infty$, $\vec{Y}(t)$ gets asymptotically close to $c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The argument we made here is true for any starting point, so we can sketch in other solutions, similarly.

Phase Portrait of Systems from Example 1 revisited

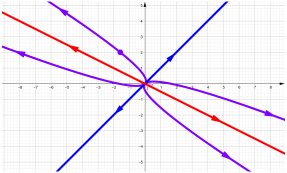
In our opening example, we found that the Genereal Solution to:

$$\vec{Y}' = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix} \cdot \vec{Y} \qquad \text{where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

in terms of constants c_1 and c_2 are: $\vec{Y}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Let's sketch the Phase Portrait using the Gen. Sol. we found.

Straight-line sol's are scalar multiples of e-vecs $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$



Since both, $e^{3t}, e^{6t} \to \infty$, these straight-line sol's move away from the origin.
What about the curves of the other solutions, where $c_1 \neq 0$ and $c_2 \neq 0$?
Starting with some generic point, what happens as $t \to \infty$?

As $t \to \infty$, $\vec{Y}(t)$ becomes a larger and larger sum of $\vec{v}_1$ and $\vec{v}_2$

But e^{6t} is increasing faster and will be the dominant term for $\vec{Y}(t)$

That is, $\vec{Y}(t)$ becomes asymptotically parallel to $\vec{v}_2$ as $t \to \infty$

As $t \to -\infty$, $\vec{Y}(t) \to \vec{0}$ since $e^{3t}, e^{6t} \to 0$, but, e^{3t} will be dominant

Solutions of Systems of Diff. Eq. Example 3

Example: Find the General Solution of the Linear System of Diff. Eq. given by:

$$\vec{Y}' = \begin{pmatrix} -5 & 2 \\ 1 & -4 \end{pmatrix} \cdot \vec{Y} \qquad \text{where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

We start by finding the e-values λ_1 and λ_2 as roots of the characteristic eq:

$$0 = det \begin{pmatrix} -5-\lambda & 2 \\ 1 & -4-\lambda \end{pmatrix} = \lambda^2 + 9\lambda + 18$$

We get the values $\lambda_1 = -3$ and $\lambda_2 = -6$ by factoring this: $0 = (\lambda+3)(\lambda+6)$
For $\lambda_1 = -3$, $\lambda_2 = -6$ we need to compute an eigenvector for each

$\lambda_1 = -3$ is an eigenvalue, so for it's eigenvector, $\vec{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} -5a+2b \\ 1a-4b \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 1 & -4 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = -3 \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -3a \\ -3b \end{pmatrix}$$

We can find a relationship between a, b from: $-5a + 2b = -3a \Rightarrow a = b$

$\lambda_2 = -6$ is an eigenvalue, so for it's eigenvector, $\vec{v}_2 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} -5a+2b \\ 1a-4b \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 1 & -4 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = -6 \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -6a \\ -6b \end{pmatrix}$$

We can find a relationship between a, b from: $-5a + 2b = -6a \Rightarrow a = -2b$

General Solution: $\vec{Y}(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Phase Portrait of Solutions from Example 3

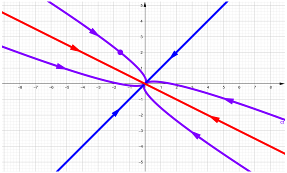
In our last example, we found that the Genereal Solution to:

$$\vec{Y}' = \begin{pmatrix} -5 & 2 \\ 1 & -4 \end{pmatrix} \cdot \vec{Y} \qquad \text{where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

in terms of constants c_1 and c_2 are: $\vec{Y}(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Let's sketch the Phase Portrait using the Gen. Sol. we found.

Straight-line sol's are scalar multiples of e-vecs $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$



Since both, $e^{-3t}, e^{-6t} \to 0$, these straight-line sol's move into the origin.
What about the curves of the other solutions, where $c_1 \neq 0$ and $c_2 \neq 0$?
Starting with some generic point, what happens as $t \to \infty$?

As $t \to \infty$, $\vec{Y}(t) \to \vec{0}$ as a smaller and smaller sum of $\vec{v}_1$ and $\vec{v}_2$

But e^{-6t} is decreasing faster so the $\vec{v}_1$ term will be the dominant term of $\vec{Y}(t)$

That is, $\vec{Y}(t)$ goes into the origin asymptotically parallel to $\vec{v}_1$ as $t \to \infty$

As $t \to -\infty$, $\vec{Y}(t)$ moves away from the origin asymptotically parallel to $\vec{v}_2$

Equilibrium Solutions of Linear Systems of Diff. Eq.

Earlier in the course, we classified equilibrium solutions of differential equations, and used the these classifications to understand the long term behavior of solutions near the equilibrium solutions.

For Linear Systems of Diff. Eq. we'll study and classify equilibrium solutions

We define an *equilibrium solution* of a system of differential equations $\vec{Y}'(t) = A\vec{Y}(t)$ to be a solution $\vec{Y}(t)$ such that $\vec{Y}'(t) = \vec{0}$.

In this case, where $\vec{Y}'(t) = \vec{0}$ we have that:

$$\vec{0} = \vec{Y}'(t) = A\vec{Y}(t)$$

and thus our equilibrium solutions satisfies $\vec{0} = A\vec{Y}(t)$ at all values of t

For any value $t = t_o$, the Linear System of equations $\vec{0} = A\vec{Y}(t_o)$ has a unique solution when $det(A) \neq 0$.

And it's easy to check that $\vec{Y}(t_o) = \vec{0}$ is a solution, and thus the unique solution.

So, for all Linear Systems of Differential Equations, $\vec{Y}'(t) = A\vec{Y}(t)$, where $det(A) \neq 0$ the only equilibrium solution is $\vec{Y} = \vec{0}$

Note 1: This equilibrium solution is the origin in our phase plane.

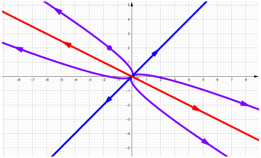
Equilibrium Solutions of Linear Systems of Diff. Eq.

The Linear system of Diff. Eq.'s $\vec{Y}' = A\vec{Y}$, with $det(A) \neq 0$, has a unique equilibrium solution, which is: $\vec{Y} = \vec{0}$

We will classify this eq. sol. based on the behavior of nearby solutions.

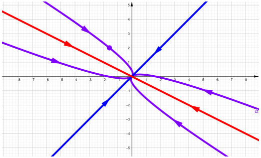
If both straight-line sol's, and thus all sol's, go out of the origin, we call the eq. sol. a *source*

$0 < \lambda_1 < \lambda_2$



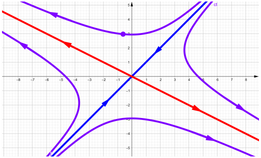
If both straight-line sol's, and thus all sol's, go into the origin, we call the eq. sol. a *node (or sink)*

$\lambda_1 < \lambda_2 < 0$



If one straight-line sol goes into and one out of the origin, we call the eq. sol. a *saddle point*

$\lambda_1 < 0 < \lambda_2$



Note: We can classify the eq. sol. from the signs of the eigenvalues alone.

Example 3 revisited with an Initial Condition

Example: Find the solution to the Initial Value Problem given by the diff. eq.:

$$\vec{Y}' = \begin{pmatrix} -5 & 2 \\ 1 & -4 \end{pmatrix} \cdot \vec{Y} \qquad \text{where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

and the Initial Condition $\vec{Y}(0) = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

We already found the General Solution to be:

$$\vec{Y}(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

It remains to find the values c_1 and c_2 using the initial condition $\vec{Y}(0) = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

Evaluating our general solution at $t = 0$ we get:

$$\begin{pmatrix} 7 \\ 1 \end{pmatrix} = \vec{Y}(0) = \cancelto{1}{c_1 e^{-3\cdot 0}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \cancelto{1}{c_2 e^{-6\cdot 0}} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 - 2c_2 \\ c_1 + c_2 \end{pmatrix}$$

So, we're left with the linear system: $7 = c_1 - 2c_2$ and $1 = c_1 + c_2$
We can solve this linear system by subtracting the first equation from the second to get: $3c_2 = -6$
So, $c_2 = -2$, which we can use to find $c_1 = 3$

Conclusion: $\vec{Y}(t) = 3e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2e^{-6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Solutions of Systems of Diff. Eq. - A Different Example

Example: Find the solution to the Linear System of Diff. Eq.:

$$\vec{Y}' = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \cdot \vec{Y} \qquad \text{where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

We start by finding the e-values λ_1 and λ_2 as roots of the characteristic eq:

$$0 = det \begin{pmatrix} -2-\lambda & -3 \\ 3 & -2-\lambda \end{pmatrix} = \lambda^2 + 4\lambda + 13$$

We cannot factor, so we'll need the quadratic formula:

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{(4)^2 - 4 \cdot 13}}{2}$$

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

How do we find a solution if the eigenvalue is complex?

Recall: For a matrix A with eigenvalue, λ, with eigenvector, $\vec{v}$, we showed that $\vec{Y} = e^{\lambda t}\vec{v}$ is a solution to $\vec{Y}' = A\vec{Y}$.

The computation to prove did not rely on λ and $\vec{v}$ being real.

So, if we can find the eigenvector for $\lambda_1 = -2 + 3i$ then we can find a solution.

Solutions of Systems of Diff. Eq. - A Different Example

$\lambda_1 = -2 + 3i$ is an eigenvalue, so for it's eigenvector, $\vec{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$$\begin{pmatrix} -2a - 3b \\ 3a - 2b \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -2 + 3i \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} (-2+3i)a \\ (-2+3i)b \end{pmatrix}$$

We can find a relationship between a, b from: $-2a - 3b = (-2+3i)a$

Adding $2a$ to both sides, gives the relationship: $-3b = 3ai$

And dividing by -3 gives that: $b = -ai$

We can get rid of the negative here by multiplying both sides by i to get: $a = bi$

So, the diff. eq. has a solution: $Y_c(t) = e^{(-2+3i)t} \begin{pmatrix} i \\ 1 \end{pmatrix}$

Notice: Our original differential equation was real but our solution is complex.

A theorem about complex solutions will help us find real solutions from this complex one.

Solutions of Systems of Diff. Eq. - A Different Example

Theorem: If $\vec{Y}_c(t)$ is a complex solution of the Linear System of Diff. Eq.:

$$\vec{Y}' = A\vec{Y}$$

and if $\vec{Y}_c(t)$ can be written in its real and imaginary parts:

$$\vec{Y}_c(t) = \vec{Y}_r + i \cdot \vec{Y}_i$$

where $\vec{Y}_r$ and $\vec{Y}_i$ are real-valued functions then $\vec{Y}_r$ and $\vec{Y}_i$ are, also, solutions.
This can be proved by comparing the real and im. parts of:

$$\left(\vec{Y}_r + i \cdot \vec{Y}_i\right)' = A \cdot \left(\vec{Y}_r + i \cdot \vec{Y}_i\right)$$

So, if we can write $Y_c(t) = e^{(-2+3i)t}\begin{pmatrix} i \\ 1 \end{pmatrix}$ in terms of its real and imaginary parts, $\vec{Y}_r$ and $\vec{Y}_i$ then these will both be solutions to our original diff. eq.
And using the two solutions, $\vec{Y}_r$ and $\vec{Y}_i$, we can build the General Solution:

$$\vec{Y}(t) = c_1\vec{Y}_r + c_2\vec{Y}_i$$

So, it remains to write $Y_c(t) = e^{(-2+3i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} = \vec{Y}_r + i \cdot \vec{Y}_i$

Solutions of Systems of Diff. Eq. - A Different Example

To write $Y_c(t) = e^{(-2+3i)t} \left(\begin{array}{c} i \\ 1 \end{array} \right) = \vec{Y}_r + i \cdot \vec{Y}_i$ in it's real and imaginary parts,

we'll start by writing each $e^{(-2+3i)t}$ and $\left(\begin{array}{c} i \\ 1 \end{array} \right)$ in their real and imaginary parts

We can easily split $\left(\begin{array}{c} i \\ 1 \end{array} \right)$ into real and imaginary parts:

$$\left(\begin{array}{c} i \\ 1 \end{array} \right) = \left(\begin{array}{c} 0 \\ 1 \end{array} \right) + \left(\begin{array}{c} i \\ 0 \end{array} \right) = \left(\begin{array}{c} 0 \\ 1 \end{array} \right) + i \cdot \left(\begin{array}{c} 1 \\ 0 \end{array} \right)$$

Now we need to write $e^{(-2+3i)t}$ in real and imaginary parts
We can split $e^{i \cdot 3t}$ into real and imaginary parts using Euler's Formula:

$$\underbrace{e^{iz}} = \underbrace{\cos(z) + i \sin(z)}$$

This gives: $e^{(-2+3i)t} = e^{-2t} \cos(3t) + i e^{-2t} \sin(3t)$

Combining these to get $\vec{Y}_c(t)$, we have:

$$\vec{Y}_c(t) = e^{(-2+3i)t} \left(\begin{array}{c} i \\ 1 \end{array} \right) = \left(e^{-2t} \cos(3t) + i e^{-2t} \sin(3t) \right) \left(\left(\begin{array}{c} 0 \\ 1 \end{array} \right) + i \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \right)$$

We will need to distribute to split $\vec{Y}_c(t)$ into real and imaginary parts.

Solutions of Systems of Diff. Eq. - A Different Example

Distributing in $\vec{Y}_c(t)$, we get

$$\vec{Y}_c(t) = \left(e^{-2t}\cos(3t) + ie^{-2t}\sin(3t)\right)\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$= e^{-2t}\cos(3t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + ie^{-2t}\cos(3t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + ie^{-2t}\sin(3t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i^2e^{-2t}\sin(3t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= e^{-2t}\cos(3t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - e^{-2t}\sin(3t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\left(e^{-2t}\cos(3t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-2t}\sin(3t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= \underbrace{\begin{pmatrix} -e^{-2t}\sin(3t) \\ e^{-2t}\cos(3t) \end{pmatrix}}_{=\vec{Y}_r(t)} + i\cdot\underbrace{\begin{pmatrix} e^{-2t}\cos(3t) \\ e^{-2t}\sin(3t) \end{pmatrix}}_{=\vec{Y}_i(t)}$$

Now, we want to separate the real and imaginary parts

Note that, despite it's name, $\vec{Y}_i(t)$ is a real function.

Both $\begin{pmatrix} -e^{-2t}\sin(3t) \\ e^{-2t}\cos(3t) \end{pmatrix}$ and $\begin{pmatrix} e^{-2t}\cos(3t) \\ e^{-2t}\sin(3t) \end{pmatrix}$ are solutions to the system

General Solution: $\vec{Y}(t) = c_1\begin{pmatrix} -e^{-2t}\sin(3t) \\ e^{-2t}\cos(3t) \end{pmatrix} + c_2\begin{pmatrix} e^{-2t}\cos(3t) \\ e^{-2t}\sin(3t) \end{pmatrix}$

Solutions of Systems of Diff. Eq. - A Different Example

Let's review the process of solving a Linear System of Differential Equation

$$\vec{Y}' = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \cdot \vec{Y} \qquad \text{where } \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

We started by finding the characteristic equation:

$$0 = det(A - \lambda I) = \lambda^2 + 4\lambda + 13$$

Solving the characteristic equation, we got $\lambda = -2 \pm 3i$

Using one of these eigenvalues, $\lambda_1 = -2 + 3i$, we found its e-vector: $\begin{pmatrix} i \\ 1 \end{pmatrix}$

With $\lambda_1 = -2 + 3i$ and $\begin{pmatrix} i \\ 1 \end{pmatrix}$ we built the complex solution:

$$Y_c(t) = e^{(-2+3i)t} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

We then needed to write $\vec{Y}_c$ in its real and imaginary parts:

$$\vec{Y}_c(t) = \vec{Y}_r + i \cdot \vec{Y}_i$$

Because then, $\vec{Y}_r$ and $\vec{Y}_i$ are both real-valued solutions, and thus:
General Solution: $\vec{Y}(t) = c_1 \vec{Y}_r + c_2 \vec{Y}_i$

To write $\vec{Y}_c$ this way, we needed split the eigenvector up into to real and imaginary parts and use Euler's Formula to get that:

$$e^{(-2+3i)t} = e^{-2t}\cos(3t) + ie^{-2t}\sin(3t)$$

Equilibrium Solutions of Linear Systems of Diff. Eq.

Recall: The Linear system of Diff. Eq.'s $\vec{Y}' = A\vec{Y}$, with $det(A) \neq 0$, has a unique equilibrium solution, which is: $\vec{Y} = \vec{0}$

In cases where the eigenvalues $\lambda_{1,2} = \gamma \pm i\mu$, the equilibrium solutions will behave differently than in the cases we saw where $\lambda_{1,2}$ are real.

If $\lambda_1 = \gamma + i\mu$ has the eigenvector, $\vec{v}$, then we'll have the solution:

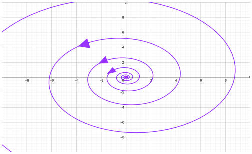
$$\vec{Y}_c(t) = e^{(\gamma + i\mu)t} \cdot \vec{v} = e^{\gamma t} \underbrace{(\cos(\mu t) + i\sin(\mu t))}_{\text{length} = 1} \vec{v}$$

The factor $(\cos(\mu t) + i\sin(\mu t))$ always has length $= 1$, and will be periodic.

Since $\vec{v}$ has constant length, the length of $\vec{Y}(t)$ varies with $e^{\gamma t}$

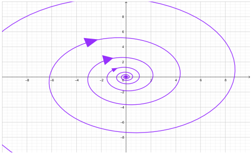
Adding the real and imaginary parts of $\vec{Y}_c$, to get a real solution, won't change the length of the solution.

So, in the phase plane, the solution will periodically spiral around the origin, with a length that varies with $e^{\gamma t}$



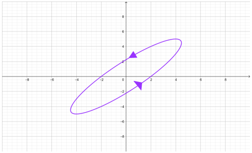
$\gamma > 0$

spiral source



$\gamma < 0$

spiral sink (node)



$\gamma = 0$

center

Equilibrium Solutions of Linear Systems of Diff. Eq.

Since solutions spiral into or away from the origin, we can consider whether the solutions are spiraling clockwise or counter-clockwise.

Due to the uniqueness of solutions to initial value problems, no two solutions can intersect and, thus, a given differential equation cannot have some solutions that spiral clockwise while other solutions spiral counter-clockwise.

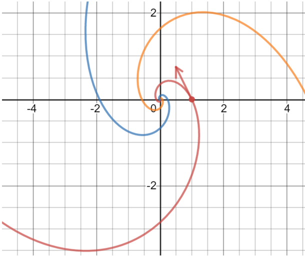
That is, for a given differential equation, all solutions must spiral clockwise or all solutions must spiral clockwise.

The direction of the spin for one solution determines the spin for all solutions.

Moreover, determining the direction of the spin at one point is sufficient.

To see how to determine this, consider our earlier example given by:

$$\vec{Y}'(t) = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \vec{Y}(t)$$



The matrix has complex eigenvalues: $-2 \pm 3i$

Since real part $-2 < 0$, the eq sol is a spiral sink.

$\vec{Y}'|_{\vec{Y}=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}$ gives the vec. tan. to $\vec{Y}$ at $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$

$$\vec{Y}'|_{\vec{Y}=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

The solution points up as it spirals into the origin, so the direction of the spin is counterclockwise.

Since one solution spins counterclockwise, all solutions spin counterclockwise.